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METRIC POSTULATES FOR PLANE GEOMETRY

SAUNDERS MacLANE, University of Chicago

1. Introduction. The current interest in better high school mathematics directs attention to Euclidean geometry, as that part of the high school curriculum with the most subtle logical structure. For this very reason, there is little agreement on the arrangement of geometry for the schools. Even in its best form, the Euclidean arrangement suffered from various imperfections centering in the neglect of betweenness and separation properties; in recent years this arrangement has undergone steady attrition from all hands—those who would transfer training and those who wouldn't; those who would use geometry as a vehicle for reasoning in everyday life and those who prepare lists of "essential theorems." It is small wonder that the idea of a logical structure for elementary geometry is sometimes lost from sight.

A break with these difficulties is provided by the *Basic Geometry* of Birkhoff and Beatley [2]. These authors observe that the real number system is at hand in the schools; assuming it, they take the measures of distance and of angle as primitives for geometry; the result is that they build up the traditional system of geometry much more quickly than did Euclid [7], Hilbert [8], or Veblen [12]. The basic reason for the speed is simple: the subtleties of the betweenness relation are mastered by the use of the order relation for numbers. The particular axioms to be used for the measures of angles and distance can be varied and perhaps improved from Birkhoff's original form [1]. The problem is easily stated: to find a simple and intuitive set of facts on distance and angle which suffice to characterize plane geometry. An efficient metric foundation of geometry in this manner would be of such manifest utility that I present herewith a statement of one such variant, in the hope that others may be encouraged to try their hands at improving this type of axiomatics.

That axioms on distance and angle measure will suffice is clear, for distance can be used to define betweenness and this latter concept alone is a sufficient primitive for geometry in Veblen's system [11]. Blumenthal [4] and Gillam [6] have also considered axiom systems based just on distance. Another such system for plane geometry appears as support for a philosophical thesis in Reidemeister [10]; however, he takes the Pythagorean theorem as an axiom, while we would rather deduce this from more evident geometric facts.

2. Primitive notions. This axiomatics applies to the plane. Primitive terms are point, distance, line, and angle measure. The distance $d(AB)$ is intended to be the usual (nonnegative) distance from the point A to the point B. Practically, it is the quantity measured by a ruler. One sees at once that the points P on the segment from A to B are exactly those points for which $d(AB)$ is the sum $d(AP)+d(PB)$. Furthermore, the points on the ray AB consist of these points P on the segment, plus the points Q for which B is on the segment AQ. In other words, segments, rays, and, for the matter of that, lines may be defined in terms of distance.

Finally, if r and s are any two rays from the same point, the angle $\angle rs$ is to be the angle measured in degrees counterclockwise from r to s. Practically, it is the quantity measured (in degrees) by a protractor. Note especially that, with Birkhoff [1], we take angles to be directed. This fact, which is not emphasized in the book [2], seems useful for this type of axiomatics, since it gives immediate access to separation properties. For example, to show that a line l separates the plane, we take any ray r on l; the side to the left of r is then the union of all these rays s such that the $\angle rs$ is less than 180°. More detail is given below, but the basic idea is clear. Directed angles are fundamental to geometry and to elementary trigonometry, which we regard as part of geometry; they are the basis of the important facts about orientation; they should appear in beginning geometry. Even the most fuzzy-minded teenager can understand that a left is a left, not a right, glove, while those to whom this example may not appeal can surely distinguish a left turn from a right one.

3. The axioms on distance. There are four axioms on points and distances, as follows.

D1. *There are at least two points.*

D2. *If A and B are points, $d(AB)$ is a nonnegative number.*

D3. *For points A and B, $d(AB)=0$ if and only if $A=B$.*

D4. *If A and B are points, $d(AB)=d(BA)$.*

Axioms D2, D3, and D4 are those for a semimetric space. If the usual triangle axiom is added, we would have the axioms for a metric space; however we do not need the triangle axiom.

A point B is said to lie *between* the points A and C if all three points are different and $d(AC)=d(AB)+d(BC)$. If $A\neq C$, the interval, Int AC, is defined to be the set of all points B which lie between A and C. Thus Int AC is intended to be the usual open interval from A to C. If $A\neq O$ are points we now define

$$\text{Ray } OA = \{P \mid P \neq O \text{ and } d(AP) = |\, d(OA) - d(OP)\,|\}.$$

Any such set $r=$ Ray OA is called a *ray* from O; thus each ray from O is a set of points not containing O. Ray OA could also be described as the union of the interval OP with the set of all points P such that either $P=A$ or A lies between O and P. For that matter, Int OA can now be described as the set of all points P on Ray OA for which $d(OP)<d(OA)$. Also, Int OA is the intersection of Ray OA with Ray AO.

4. The axioms on lines. There are four axioms on lines, as follows:

L1. *A line is a set of points containing more than one point.*

L2. *Through two distinct points there is one and only one line.*

L3. *Three distinct points lie on a line if and only if one of them is between the other two.*

L4. *On each ray from a point O and to each positive real number b there is a point B with $d(OB)=b$.*

For distinct points $A \neq B$, we shall denote by Line (AB) the unique line l containing A and B. In view of Axiom L3 this line is the union of the following four disjoint sets: Int (AB); the set of all P with A between P and B; the set of all Q with B between A and Q, and the set consisting of A and B. In particular, Ray $AB \subset$ Line AB.

LEMMA 1. *If A, B, and C are collinear, $d(AC) \leqq d(AB)+d(BC)$.*

This asserts that the triangle axiom holds at least for the points of any one line. The proof is direct from Axiom L3. Either B is between A and C, in which case the equality holds, or one of A or C lies between, in which case the inequality applies.

LEMMA 2. *If B is between O and A, then* Int $OB \subset$ Int OA.

Proof. Take any $P \in$ Int OB; then $d(OB)=d(OP)+d(PB)$. By hypothesis, B is between O and A, so $O \neq A$ and $d(OA)=d(OB)+d(BA)$. Combining these equations and applying the triangle axiom (which holds because all points concerned lie on Line (OA)) we have

$$d(OA) = d(OP) + d(PB) + d(BA) \geqq d(OP) + d(PA).$$

But the triangle axiom also gives the reverse inequality $d(OA) \leqq d(OP)+d(PA)$. Hence equality holds, and P is between O and A.

LEMMA 3. *On each ray from a point O and to each positive real number b there is exactly one point B with $d(OB)=b$.*

Proof. Suppose, instead, that there were two such points B and B' on some Ray OA. Then O, B, B', and A all lie on a line, to wit, the line OA. For the purposes of this proof let us write $[x, y, z]$ when the three real numbers x, y, and z are such that one of them is the sum of the other two. Then O, B, B' collinear implies by Axiom L3 that $[b, b, d(BB')]$; since $B \neq B'$ and $d(BB') \neq 0$, this gives $d(BB')=2b$. Set $a=d(OA)$.

Case 1: $b>a$. Then A is between O and B and also between O and B'; hence $d(AB)=b-a=d(AB')$. Then A, B, B' collinear gives $[b-a, b-a, d(BB')]$; hence $d(BB')=2b-2a$. Since $a \neq 0$, this contradicts the previous value $d(BB')=2b$.

Case 2: $b=a$. Then $B \in$ Ray OA gives either $B \in$ Int OA or $A \in$ Int OB. If the former, then $a=d(OA)=d(OB)+d(BA)$, whence $d(BA)=0$ and $B=A$ by Axiom D3. The same conclusion holds in the latter case. All told, $B=A=B'$, in contradiction to the assumption $B \neq B'$.

Case 3: $b<a$. Here B and B' are both between O and A, so that $d(BA)=a-b=d(B'A)$. Then A, B, B' collinear gives $[a-b, a-b, d(BB')]$; hence $d(BB')=2a-2b$. The previous value was $d(BB')=2b$. Hence $a=2b$ and $d(OB)=d(BA)=d(AB')=d(B'O)=b$, $d(OA)=d(BB')=2b$. In other words, O, B, A, B' are, in order, the vertices of a "square" with sides b with both "diagonals" of length $2b$.

This "square" has the property that any *three* of its points can be embedded in the ordinary real line, preserving distance, while all four cannot be so embedded. Such a configuration is known in distance geometry (*cf.* [4]) as a pseudo-Euclidean quadruple.

To show this impossible, we use Axiom L4 to choose a point C on Ray OA with $d(OC)=3b$. Then A is between O and C, and B, $B'\in$Int $OA\subset$Int OC, by Lemma 2. Hence $d(BC)=2b=d(B'C)$. Then B, B' and C collinear yields $[2b, 2b, 2b]$, a contradiction.

This completes the proof in all cases.

THEOREM 1. *If* $O\neq A$ *and* $B\in$Ray OA, *then* Ray $OB=$Ray OA.

In proving this we may assume $B\neq A$. Then $B\in$Ray OA asserts that either $B\in$Int OA or $A\in$Int OB. Since our conclusion is symmetric in A and B, it will suffice to prove, say, that $B\in$Int OA implies Ray $OB=$Ray OA. Also, Ray OB contains a point at any given distance from O, while by Lemma 3, Ray OA contains only one such point. Hence it will suffice to prove that

$$B \in \text{Int } OA \quad \text{implies} \quad \text{Ray } OB \subset \text{Ray } OA.$$

To show this, take $P\in$Ray OB and consider the three possible cases for P. If $P\in$Int OB, then by Lemma 2, Int $OB\subset$Int $OA\subset$Ray OA, and so $P\in$Ray OA. If $P=B$, then $P\in$Ray OA. There remains the case when $B\in$Int OP, so that $d(OP)=d(OB)+d(BP)$. Since $B\in$Int OA, we also have $d(OA)=d(OB)+d(BA)$. Suppose now that $P\in$Ray OA is false; since P is on the line OA we then have O between P and A and hence

$$\begin{aligned} d(PA) &= d(PO) + d(OA) \\ &= d(OB) + d(BP) + d(OB) + d(BA) \\ &\geqq 2d(OB) + d(PA), \end{aligned} \qquad \text{(Lemma 1)},$$

a manifest contradiction to $d(OB)\neq 0$.

COROLLARY. *Any two rays from a fixed point O are equal or disjoint.*

This is immediate.

For elementary instruction some of these arguments, notably that for Lemma 3, will seem too artificial. This can be handled by making Lemma 3 a part of Axiom L4. Going further, we can add Theorem 1 as an axiom. From this Lemmas 2 and 3 can be proved easily. For example, to prove Lemma 3 observe

that Theorem 1 gives Ray OA = Ray OB, so that the proof of Lemma 3 need only be given in Case 2.

5. Coordinates on a line.

THEOREM 2. *If O is any point of the line l, then l contains exactly two disjoint rays from O, and every point $B \neq O$ on l is on one of these rays.*

Proof. By Axiom L1 there is on l a point $A \neq O$. By Axiom L4, there is on Ray AO a point A' with $d(AA') = 2d(OA)$. Since A' cannot be between O and A, we must then have O between A and A'. If $a = d(OA)$, then $d(AA') = 2a$. Let r = Ray OA, r' = Ray OA'. Take $B \neq O$ on l. If B is not in r, then by Axiom L3 and the definition of a ray, O is between A and B, and $d(AB) = a + d(OB)$. If also B is not in r', then O is between A' and B, and $d(A'B) = a + d(OB)$. But A, A', and B are collinear and distinct, which implies that one of the three numbers $a + d(OB)$, $a + d(OB)$, and $2a$ is the sum of the other two, which is impossible. Therefore B is either on r or on r'. Since $A' \in r'$ and $A' \notin r$, Theorem 1 shows that r and r' can have no points in common. Any ray from O contained in l must have a point on one of r or r'; hence by Theorem 1 it must be either r or r'. We may call r' the ray *opposite* r.

We can now define coordinates on a line l. If $O \in l$, choose one of the two rays from O in l, and call it the "positive" ray r; call the other ray from O in l the "negative" ray r'. Define a function x on the set l to the real numbers by setting $x(O) = 0$, $x(P) = d(OP)$, if $P \in r$, and $x(P) = -d(OP)$ if $P \in r'$. We call x the *coordinate function* on l with origin O and positive direction r. One readily proves

THEOREM 3. *Each coordinate function x on the line l is a one-one mapping of l onto the set of real numbers such that $d(PQ) = |x(P) - x(Q)|$ for all P, Q on l.*

Birkhoff assumed as an axiom the existence of a coordinate function with the property stated in Theorem 3. Since such a coordinate function is by no means unique, we have regarded it as appropriate to analyze his axiom into more primitive assertions about distance as formulated in our axioms above.

From this theorem it follows that Q lies between P and R if and only if either $x(P) < x(Q) < x(R)$, or $x(P) > x(Q) > x(R)$. Hence, also, the two rays from any point A on l are the sets of all P with $x(P) > x(A)$ and $x(P) < x(A)$, respectively. It follows that if x and y are any two coordinate functions on l, there is a real number a and a number $\epsilon = \pm 1$ such that $x(P) = \epsilon y(P) + a$ for all $P \in l$. The choice $\epsilon = -1$ amounts to a change in the "direction" of the line.

The notion of directions can be treated formally if one wishes. A *direction* D on a line l can be described as a function which assigns to each point $P \in l$ a ray $D(P)$ from P and contained in l, in such fashion that for any two points P and Q in l, the intersection $D(P) \cap D(Q)$ is one of $D(P)$ or $D(Q)$. We may then say that P comes *before* Q in the direction D if $D(P) \cap D(Q) = D(Q)$. This relation "before" yields a transitive order of the points on the line. Using the results above, one may easily prove

THEOREM 4. *A line has exactly two directions D and D'.*

Given one of these directions D, the second is described by letting $D'(P)$ be the ray opposite $D(P)$ at every point P. A direction is determined by any one of its positive rays; that is, given any ray r from a point O contained in the line l, there is exactly one direction D of l with $D(O)=r$.

6. One-dimensional geometry. The discussion hitherto has been essentially one-dimensional. It yields a set of metric axioms sufficient to characterize the one-dimensional Euclidean space: the Axioms D1, D2, D3, D4, the definition of "betweenness" and of "ray," the Axiom L4, and a rephrased Axiom L3:

L3′. *Given three distinct points, one of them lies between the other two.*

There are many other ways of setting up such axiom systems for a line; for example, one is given by Reidemeister in [10].

7. The axioms on angles. We regard an angle as the figure (or ordered triple) consisting of a point O and two rays r and s from O. Since we measure angles in degrees, it is convenient for each real number c to write $c°$ for the number c taken modulo 360.

There are four axioms on angle measure.

A1. *If r and s are rays from the same point, $\angle rs$ is a real number modulo* 360.

A2. *If r, s, and t are three rays from the same point, then $\angle rs+\angle st=\angle rt$.*

A3. *If r is a ray from O and c is a real number, then there is a ray s from O such that $\angle rs=c°$.*

If $A\neq O\neq B$ are points, we write $\angle AOB$ as usual for $\angle$(Ray OA)(Ray OB). Then

A4. *If $A\neq O\neq B$, then $\angle AOB=\angle BOA\neq 0°$ if and only if $d(AB)=d(AO)+d(OB)$.*

There are only two more axioms to come later, a *similarity* axiom and a *continuity* axiom. For the moment we observe that there are three existence axioms: the existence of points (D1); of points on a ray (L4); and of rays on a point (A3).

Axiom A2 giving the additivity of angles is possible only because we deal with directed angles. It is a most useful axiom; for example, it implies at once that $\angle rr=0$ and that $\angle rs=-\angle sr$. In Axiom A4, observe that always $\angle AOB+\angle BOA=360°=0°$; hence $\angle AOB=\angle BOA\neq 0°$ implies that $\angle AOB=180°$, so the axiom is equivalent to the following requirement;

A4′. *If $A\neq O\neq B$, then $\angle AOB=180°$ if and only if O is between A and B.*

THEOREM 5. *Let r and s be two rays from the same point O. If $\angle rs=180°$, then r and s, together with O, form a straight line. If $\angle rs=0°$, then $r=s$.*

Proof. Suppose first that $\angle rs = 180°$; then $r \neq s$ and hence $r \cap s = 0$. Pick points A on r and B on s; then $A \neq O \neq B$, and $\angle AOB = \angle rs = 180°$; hence, by Axiom A4, O lies between A and B and therefore on the line AB. This line contains r and s; by Theorem 2 it must consist of r, s, and O, as asserted.

Suppose next that $\angle rs = 0°$. By Axiom A3 there is a ray t from O with $\angle st = 180°$; hence s, t, and O form a line. But then $\angle rt = \angle rs + \angle st = 180°$, so that again r, t, and O form a line. It is the same line, hence r is one of s and t, by Theorem 2. Since $r \neq t$, we get $r = s$, as asserted.

COROLLARY. *If $A \neq O$, Ray OA consists of all points $B \neq O$ with $\angle AOB = 0°$.*

We can now introduce polar coordinates with O as origin and prove the usual facts about the one-one correspondence between points and their polar coordinates.

8. Similar triangles. With Birkhoff we deduce the theory of parallel lines from that of similar triangles. One reason for this order of events is that the classical order, with parallels first, involves a theorem that an exterior angle of a triangle exceeds either remote interior angle; the usual proof makes tacit assumptions about sides of a line.

By an *ordered triangle* we mean the figure $\triangle ABC$ formed by three distinct points A, B, and C; if you wish, the three intervals which they determine can be regarded as part of the figure. But notice that we name the vertices in order. This saves trouble, for the same reason that the use of ordered simplices in the modern version of singular homology theory saves trouble (see, *e.g.*, S. Eilenberg, *Singular Homology Theory*, Ann. of Math. vol. 45, 1944, pp. 407–447). A less sophisticated reason is better: ordered triangles are what actually occur in the theorems of elementary geometry.

Two ordered triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar if there is a positive real number k and number $\epsilon = \pm 1$ such that $\angle ABC = \epsilon \angle A'B'C'$, $\angle BCA = \epsilon \angle B'C'A'$, $\angle CAB = \epsilon \angle C'A'B'$, $kd(AB) = d(A'B')$, $kd(BC) = d(B'C')$, $kd(CA) = d(C'A')$. The triangles are congruent if this holds with $k = 1$. We assume the side-angle-side (SAS) case of similarity as an axiom, in the following form.

SIMILARITY AXIOM. *If two ordered triangles $\triangle ABC$ and $\triangle A'B'C'$ have $\angle ABC = \epsilon \angle A'B'C'$, $d(AB) = kd(A'B')$, and $d(BC) = kd(B'C')$ (for $\epsilon = \pm 1$, k positive) they are similar.*

Recall that Hilbert [8] assumed a weaker version of the congruence case ($k = 1$) of this axiom. We assume the case of similarity as well, so as to avoid the sophistication of an "incommensurable case" argument (*i.e.*, a continuity argument). This stronger assumption is in keeping with standard practice; for example, in a real vector space the distributive law $k(V + W) = kV + kW$ for multiples by a real scalar could be proved by continuity from the laws of addition, but we usually simply assume the general law.

Reinhold Baer has pointed out to me that the similarity axiom may be

interpreted in terms of "motions." Take $\epsilon = \pm 1$, $k > 0$, and let two rays r = Ray BC, r' = Ray $B'C'$ be given. There is then a "motion" M which takes r into r' with scale factor k; explicitly, M is defined as that transformation of the plane which takes each point P into the point $P' = M(P)$ with $\angle PBC = \epsilon \angle P'B'C'$ and $d(P'B') = kd(PB)$; the Axioms A3 and L3 and Theorems 1 and 5 insure that there is a unique such point P'. The similarity axiom is then the assertion that M multiplies distances by k and angles by ϵ: $d(M(P)M(Q)) = kd(PQ)$, $\angle M(P)M(O)M(Q) = \epsilon \angle POQ$. By representing this motion M as the composite of several simpler motions, one can deduce the general similarity axiom from the following three special cases: *congruence* ($k = 1$, $\epsilon = \pm 1$, $r \neq r'$), *reflection* ($k = 1$, $\epsilon = -1$, $r = r'$), and *stretch* ($k \neq 1$, $\epsilon = 1$, $r = r'$).

A triangle $\triangle ABC$ is said to be *degenerate* if its vertices A, B, and C are collinear. It suffices to postulate the similarity axioms for a nondegenerate triangle, since in the case of a degenerate triangle the axioms can be proved from the previous axioms.

From the similarity axiom one now derives the other congruence and similarity theorems in the following order. First ASA (angle-side-angle) for a nondegenerate triangle, then base angles in an isosceles triangle are equal, then SSS (three sides given), and finally the theorem that the sum of the angles in a triangle is 180°. Note that ASA may be false for a degenerate triangle. The details are as in Birkhoff [1]; note especially the elegant proof of the angle-sum proposition made by bisecting the sides of the given triangle. Each similarity theorem (SAS, ASA, *etc.*) includes the corresponding congruence theorems, though for elementary purposes it may be better to first prove the congruence theorems and then the corresponding ones on similarity.

9. The continuity axiom. Let X move along the line AB, and let O be on one side of AB. Our continuity axiom is to assert that $\angle AOX$ is a monotonic and continuous function of $x = d(AX)$. This can be put in more elementary form by speaking of "proper" angles—an angle $c°$ is *proper* if $0 < c < 180°$, and *improper* if $180° < c < 360°$. Clearly the supplement of a proper angle is proper, and the (additive) inverse of a proper angle is improper.

CONTINUITY AXIOM. *Let $\angle AOB$ be proper* (Fig. 1). *If D is between A and B, then $0 < \angle AOD < \angle AOB$. Conversely, if $0 < \angle AOC < \angle AOB$, then the ray OC meets the interval AB.*

The statement that $\angle AOX$ is continuous (as above) follows easily; it is indeed a very simple and illuminating exercise in elementary epsilontics. Birkhoff takes this continuity as an axiom (part of his Postulate III), though it is not quite clear whether he means to assume just continuity or continuity and monotonicity. In the book [2], "continuous" is defined to mean continuous and monotonic, and it is then assumed that $\angle AXB$ varies "continuously" when X varies along a line *or* a curve. This is troublesome, for if A and B are the endpoints of the diameter of a circle and X moves along a curve crossing the cir-

cumference of that circle several times, $\angle AXB$ does not vary monotonically.

We now sketch briefly how the continuity axiom gives the separation of the plane by lines. First observe the cyclic order of angles in a triangle, as in:

THEOREM 6. *If an ordered triangle $\triangle ABC$ has one angle $\angle ABC$ proper, then the remaining angles $\angle BCA$ and $\angle CAB$ are proper.*

For school use this might well be an axiom; the following proof, though simpler than that presented by Birkhoff, is still somewhat subtle. Take X on $\{B\} \cup$ Ray BC (Fig. 2) and set $x = d(BX)$. Now set $f(x) = 180° - \angle ABC - \angle XAB$. For all $x \geqq 0$, $f(x)$ is a continuous function. For $x > 0$, the angle-sum theorem gives $f(x) = \angle BXA$, which is neither 0° nor 180°. For $x = 0$, $f(0) = 180° - \angle ABC$; hence is proper. Thus $f(x)$ is a continuous function on the segment $0 \leqq x \leqq d(BC)$; it is never 0° nor 180°; therefore $f(x)$ is always proper; therefore $f(d(BC)) = \angle BCA$ is proper.

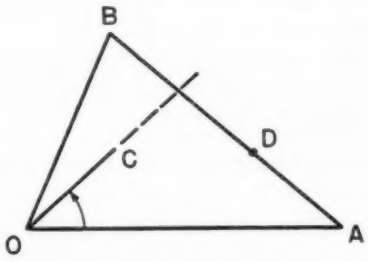

FIG. 1

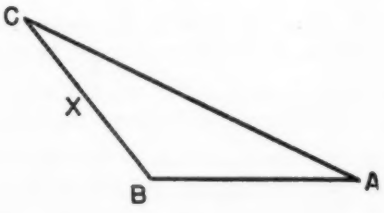

FIG. 2

We can now define the "sides" of a line l. Take a direction D on the line l, as described in Theorem 4. If A is before B in the direction D, we say that P is to the *left* of l, as directed by D, if $\angle BAP$ is proper, and to the *right* of l if $\angle BAP$ is improper. A simple application of Theorem 6 shows that this definition is independent of the choice of the points A and B; that is, if also A' is before B' in the direction D, then $\angle B'A'P$ is proper precisely when $\angle BAP$ is proper. We now can show that each line separates the plane into its left and right side.

THEOREM 7. *If l is a directed line, then points P and Q not on l lie on the same side of l if and only if l does not meet the interval PQ.*

Proof. Suppose first that $l \cap \text{Int } PQ \neq 0$, and that P is, say, to the left of l in the direction D. We must then show Q to the right of l. Let A be the point in which l meets Int PQ, and take some $B \in l$ with A before B. Then $\angle BAP$ is proper. But by Axiom A4, $\angle QAB + \angle BAP = 180°$; hence $\angle QAB$ is proper and therefore $\angle BAQ = -\angle QAB$ is improper. Hence Q is to the right of l in the direction D.

Suppose next that $l \cap \text{Int } PQ = 0$, and that P is again to the left of l. We must show that also Q is to the left of l. Again, take A before B on l in the direction D; then $\angle BAP$ is proper. For $X \in \text{Int } PQ$ the angle $\angle BAX$ is a continuous function of $d(PX)$; since $l \cap \text{Int } PQ = 0$, this function is never 0° nor 180°. Hence $\angle BAX$ is always proper; in particular, $\angle BAQ$ is proper and Q is also to the left of l.

COROLLARY (Axiom of Pasch). *Given $\triangle ABC$ and a line l with $A \notin l$, if l meets* Int BC, *then l also meets one of* Int AB *or* Int AC.

Proof. Since l meets Int BC, B and C lie on opposite sides of l. Since $A \notin l$, A lies on one of these sides, say with C. Thus A and B are on opposite sides, so that by the theorem, l meets Int AB.

10. Parallels and perpendiculars. It is convenient to start with the existence of a triangle for given ASA.

THEOREM 8. *Given an interval AB and two numbers c and d between* 0 *and* 180 *such that $c + d < 180$, there exists a point C such that the ordered triangle ABC has $\angle CAB = c°$, $\angle ABC = d°$.*

Theorem 5 shows that the ray making a given angle with a given ray is unique; hence the conclusion of this theorem could be rephrased as follows.

On the interval AB (Fig. 3), *erect at B a ray BE' with $\angle ABE' = d°$ and at A, a ray AE with $\angle EAB = c°$. The rays BE' and AE must then meet.*

Proof. We first show that there exists some triangle with angles c and d. Set $e = 180 - c - d$ and construct any triangle $\triangle LMN$ with $\angle LMN = c°$. If $\angle MNL = d°$, we are done. If $\angle MNL > d°$, the continuity axiom provides a point $X \in \text{Int } LM$ with $\angle MNX = d°$, and $\triangle MNX$ is the desired triangle with angles c and d. Otherwise $\angle MNL < d°$; the angle-sum proposition then shows that $\angle NLM > e°$. In this case the continuity axiom provides a point Y on Int MN with $\angle YLM = e°$, hence with $\angle MYL = 180° - e° - c° = d°$. In any case, we then have a triangle RST with $\angle TRS = c°$, $\angle RST = d°$.

Now return to the given interval AB. By polar coordinates, there is a point C with $\angle ABC = d°$ and $d(BC)/d(BA) = d(ST)/d(SR)$. Then $\triangle ABC$ is similar to $\triangle RST$ by the axiom of similarity; hence $\triangle ABC$ has the desired angles.

Perpendicular lines are defined as usual. Axiom A3 includes the fact that one

can erect a perpendicular to a given line at any point of that line. We also prove

THEOREM 9. *If $C \notin l$, there is through C a unique line $\perp l$.*

Proof. That there is at most one such perpendicular follows from the angle-sum proposition for a triangle. To construct one, direct l from A to B so that C lies to the left of l. Then $\angle BAC$ is proper. If $\angle BAC = 90°$, we are done. Otherwise take $B' \in l$ so that ray AB is opposite ray AB'; then one of the angles $\angle BAC$, $\angle CAB'$ is acute. If $\angle BAC$ is acute, Theorem 10 gives a triangle EAC with $\angle EAC = \angle BAC$, $\angle ACE = 90° - \angle BAC$. Then $\angle CEA = 90°$, and CE is the desired perpendicular. If $\angle CAB'$ is acute, there is a similar construction.

Two lines are said to be *parallel* if they do not meet.

THEOREM 10. *If $C \notin l$, there is through C a unique line parallel to l.*

Proof. By Theorem 9, take m through C with $m \perp l$; then take a line n through C with $n \perp m$. By the angle-sum proposition n and l cannot meet; by Theorem 8 any other line through C will meet l.

The Theorem of Pythagoras now follows, say, by dropping a perpendicular from the vertex of the right angle to the hypotenuse, and using similar triangles. With this, we can introduce rectangular coordinates, prove the distance formula, and the usual description of straight lines by linear equations. This shows that the plane described by our axioms is the classical one. The trigonometric functions may also be defined. The fact that our angle measure agrees with the classical one rests on the usual proportionality between inscribed angles and arc; the proof, as in Birkhoff [1], employs the continuity axiom.

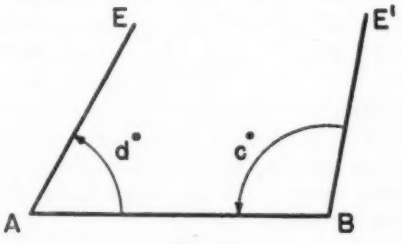

FIG. 3

11. Problems and alternatives. Our axioms suffer from at least one blemish; in pure Euclidean geometry, distance should be a magnitude with no fixed unit of measure. Such a notion of magnitude can be easily described axiomatically, but is probably not suitable for high schools; at any rate, we can easily observe that all our constructions and theorems are unaltered under change of scale (*i.e.*, when all distances involved are multiplied by some fixed positive real number). A similar remark applies to a change in the unit of angle measure.

More important is the observation that our axioms apply to an *oriented*

plane, hence cannot at once be extended to three-space. This is because there is no continuous way of choosing directions for all angles $\angle rs$ in three-space so that the addition axiom A2 will hold. It would be interesting to have an axiom system of this general type for the unoriented plane and the extension of this axiom system to space. This would allow easy comparison with the ordinary axiom systems for geometry, which are normally formulated for three dimensions.

We have not studied the independence of all of our axioms, but we can present examples to show the independence of the similarity axioms and of the continuity axiom.

To show the independence of the similarity axiom use the example of [8], page 47. Take the points, lines, and angle measures to be the usual points, lines, and angle measures of a plane II, situated in three-space, but take $d(AB)$ to be the Euclidean distance $d(A'B')$, where A' and B' are the respective projections of A and B on some other plane II′ cutting II at an acute angle. It is then easily seen that all the axioms except the similarity axiom hold. In view of this example it is convenient to speak of a system of points, lines, distances, and angle measures which satisfy the axioms D1—D4, L1—L4, and A1—A4 as a *pre-Euclidean plane.*

To show the independence of the continuity axiom use a Hamel basis of the real numbers to construct an automorphism ϕ of the additive group of real numbers such that $\phi(a)=a$ for every rational number but $\phi(b)=3b$ for some irrational number b. Now take the points, lines, and distances to be the points, lines, and distances of an ordinary plane, but define a new angle measure by $\alpha(rs)=\phi(\angle(rs))$ for any two rays r and s from the same point. Since ϕ is the identity on rational numbers, the new angle measure α has $\alpha(rs)=180 \pmod{360}$ if and only if $\angle rs=180°$. Hence the elementary axioms, and especially Axioms A1–A4 on angle measure, are still valid. That the similarity axiom continues to hold follows from the observation that two angles are congruent in the old measure if and only if they are congruent in the new measure. However, the continuity axiom is obviously violated because sometimes $\phi(b)=3b$.

Many variants of the axioms are possible. Birkhoff's system consisted essentially of the following axioms: D1–D4, L1, L2, A1, the similarity and continuity axioms, and the following two:

AXIOM L. *On each line l there is a function x which is a one-one correspondence between the points P on l and the real numbers, such that $d(PQ)=|x(P)-x(Q)|$ for all P and Q on l.*

AXIOM A. *For each point O there is a function θ which is a one-one correspondence between the rays r from O and the real numbers modulo* 360 *such that $L(rs)=\theta(s)-\theta(r)$ for all rays r and s from O.*

These axioms are exactly Birkhoff's four postulates (he used D1–D4, L1, and A1 without numbering them) with the unessential changes that his angles are measured mod 2π, and that his primitive for angles is $\angle AOB$ and not $\angle rs$

(while his rays are defined as half-lines). Since both axiom systems give the (cartesian) plane, his system is equivalent to ours; however it is not until the Theorem of Pythagoras is proved that it follows that his line AB contains all points metrically between A and B.

In our system the line need not be taken as a primitive concept, since one can define Line AB as the set of all points C such that $C=A$, $C=B$, or one of A, B, and C is between the other two. This definition does not seem to allow us to reduce the number of axioms.

Another variant of our system without lines as primitive notion is the following. Take, point, distance, and angle measure $\angle rs$ as primitive, with rays defined as above. As axioms take D1–D4, A1–A4, and D5 (the present Theorem 1) and D6 (the present L4). Then define a line to be the union of a point O with two rays r, r', from O such that $\angle rr'=180°$. It can be shown that these axioms also yield a pre-Euclidean plane.

Still other variants can be set up. For example, one may replace the angle between two rays r and s by the angle $\angle AOB$ given for any three points $A\neq O\neq B$. The ray OA can then be defined as the set of all points $B\neq O$ with $\angle AOB=0°$. As axioms for a pre-Euclidean plane, one may then take D1-D4, L4 (with uniqueness required for the point B) and the following:

A1′. *If* $A\neq O\neq B$, *then* $\angle AOB$ *is a real number modulo* 360.

A2′. *If* A, B, *and* C *are all different from* O, *then* $\angle AOB+\angle BOC=\angle AOC$.

A3′. *If* $A\neq O$, *and* c *is a real number, there is a point* $B\neq O$ *with* $\angle AOB=c°$.

A4′. *If* $A\neq O\neq B$, *then* $\angle AOB=180°$ *if and only if* $d(AB)=d(AO)+d(OB)$.

A5. *If* $A\neq O\neq B$, *then* $\angle AOB=0°$ *if and only if* $d(AB)=|d(AO)-d(BO)|$.

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AN INTEGER CONSTRUCTION PROBLEM

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1. The problem. "With the digits 1, 2, 3, 4 express the consecutive integers from 1 upwards as far as possible: each of the four digits being used once, and only once, in the expression of each number."

The puzzle quoted above from Ball* is illustrative of a common type of mathematical recreation: a numerical construction problem. A set of numbers is given (with possibly a fixed number of repetitions for each of the numbers); certain numerical operations are allowed to be performed on these numbers; and some or all of the resulting numbers are required to be found. In the puzzle quoted above, an answer is that, if only the operations of addition, subtraction, and multiplication on the integers 1, 2, 3, 4 (each integer being used exactly once) are allowed, then all integers from 1 through 28, as well as 30, 32, and 36, but no others, can be constructed. A proof of this statement is found in Section 6. If other operations are allowed, then it may be expected that different sets of numbers may be constructed; Ball gives several examples of this.

In this paper, an explicit expression is determined for a specific integer, namely the largest possible integer, constructible from a given set of positive integers, using only the operations of addition, subtraction, and multiplication. Use of the specific properties of a system of notation for integers is not allowed.

2. Solution. At first, only addition and multiplication are considered to be allowed operations. It will be shown in Theorem 2 that subtraction is never necessary.

THEOREM 1.† *Let r, s, and t be nonnegative integers, not all zero. Let r 1's, s 2's, and t integers greater than 2 be given, where the larger numbers are given in nondecreasing order:*

$$3 \leqq a_1 \leqq \cdots \leqq a_t.$$

Then the largest number constructible out of the $r+s+t$ given numbers, using addition and multiplication only, each of the $r+s+t$ integers being used exactly once, is

(a) 1, *if $r = 1$ and $s = t = 0$;*

(b) $(a_1 + 1) \cdot \prod_{j=2}^{t} a_j$ *if $r = 1$, $s = 0$, and $t \geqq 1$;*

(c) $2^{s-r} \cdot (2 + 1)^r \cdot \prod_{j=1}^{t} a_j$ *if $r \leqq s$;*

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* W. W. R. Ball, Mathematical Recreations and Essays, New York, 1947, p. 15.

† I wish to thank the referee for a strong brevity-inducing transformation of the proof of Theorem 1.

(d) $(1+1)\cdot 2\cdot(2+1)^{s-1}\cdot \prod_{j=1}^{t} a_j$ *if* $r-s=1$ *and* $s \geqq 1$;

(e) $(1+1)^2\cdot(1+1+1)^{(r-s-4)/3}\cdot(2+1)^s\cdot \prod_{j=1}^{t} a_j$

if $r-s \geqq 4$ *and* $r-s \equiv 1 \pmod 3$;

(f) $(1+1+1)^{(r-s)/3}\cdot(2+1)^s\cdot \prod_{j=1}^{t} a_j$ *if* $r>s$ *and* $r-s \equiv 0 \pmod 3$;

(g) $(1+1)\cdot(1+1+1)^{(r-s-2)/3}\cdot(2+1)^s\cdot \prod_{j=1}^{t} a_j$ *if* $r>s$ *and* $r-s \equiv 2 \pmod 3$.

If q denotes the remainder on division of $s-r$ by 3, i.e., if

$$q = s - r - 3\cdot\left[\frac{s-r}{3}\right],$$

then forms (e), (f), *and* (g) *may be written uniformly as*

(h) $(1+1)^q\cdot(1+1+1)^{(r-s-2\cdot q)/3}\cdot(2+1)^s\cdot \prod_{j=1}^{t} a_j$ *if* $r \geqq s+2$.

In the above formulas, empty products, such as $\prod_{j=2}^{1} a_j$, are considered to have the value 1. Each of the above forms indicates the explicit method of constructing a maximizing combination. The maximum value is a product whose factors are the numbers $a_1, \cdots, a_t$ and as many 3's as possible (made up out of 1's and 2's), together with any leftover 2's. A single remaining 1 is added to the smallest factor.

The forms (a) through (g) (or (a) through (d), and (h)) are mutually exclusive and exhaustive.

3. Definitions. A *primitive symbol* is a symbol chosen from the r 1's, or the s 2's, or from $a_1, \cdots, a_t$.

A *subcombination* is a nonempty sequence of symbols containing a subset of the $r+s+t$ primitive symbols once each, in any order, interspersed with $+$, $\cdot$, (, and) in such a way that the symbol sequence makes sense and can be evaluated arithmetically. The result of the evaluation is called the *value* of the subcombination. A *combination* is a subcombination using all $r+s+t$ of the primitive symbols.

4. Proof of Theorem 1.

LEMMA 1 (the product-sum form). *If X is any combination, then there is a combination Y of not smaller value such that Y is in product-sum form, i.e., Y is a product of subcombinations each of which is a sum of primitive symbols.*

In order that no errors of association shall be made, it is assumed that suffi-

ciently many parentheses are used. This can be assured by writing the result of each binary operation in parentheses.

Proof. The proof is by mathematical induction. Now each primitive symbol alone forms a subcombination in product-sum form. Suppose, then, that every subcombination using fewer than k of the primitive symbols may be replaced by a subcombination in product-sum form using exactly the same primitive symbols and having a value at least as large. Let now V be a subcombination using k of the primitive symbols, where $k>1$. Then either $V=(V_1+V_2)$ or $V=(V_1\cdot V_2)$, where each of V_1, V_2 is a subcombination using fewer than k primitive symbols. But, by the induction assumption, V_1 and V_2 are replaceable by W_1 and W_2, where W_1 and W_2 are subcombinations in product-sum form using the same primitive symbols as V_1 and V_2, respectively, and such that $V_1\leqq W_1$ and $V_2\leqq W_2$.

If $V=(V_1\cdot V_2)$, then $(W_1\cdot W_2)$ uses the same primitive symbols as V; furthermore, obviously, $V\leqq(W_1\cdot W_2)$. Hence $(W_1\cdot W_2)$ is the required subcombination in product-sum form.

Let now $V=(V_1+V_2)$. If both W_1 and W_2 have values at least 2, then $V\leqq(W_1+W_2)\leqq(W_1\cdot W_2)$, so that $(W_1\cdot W_2)$ is the required subcombination. But if, for example, W_1 has value 1, then $W_1=(X_1\cdot 1)$, where X_1 is a (possibly empty) product of 1's. Since W_2 is in product-sum form, let X_2 be one of its factors (a sum of primitive symbols), and let X_3 be the (possibly empty) product of the remaining factors of W_2. Then $V\leqq(W_1+W_2)=((X_1\cdot 1)+(X_2\cdot X_3))\leqq(X_1\cdot((1+X_2)\cdot X_3))$, since X_1 and X_3 have values at least 1. But the last expression uses exactly the same primitive symbols as V, and is also in product-sum form, since no multiplication is involved in the factor $(1+X_2)$. Thus V may be replaced by a subcombination in product-sum form if W_1 has value 1. A similar proof holds if W_2 has value 1.

The proof is completed by letting $k=r+s+t$.

LEMMA 2. *If X is a subcombination in product-sum form, then a subcombination in product-sum form can be found which has a not smaller value, which uses the same primitive symbols as X, and whose only factors are subcombinations of the form* 1, (1+1), (1+1+1), 2, (2+1), a_j, *or* (a_j+1) *for* $1\leqq j\leqq t$.

Proof. If any factor of X is expressible as a sum of two or more subcombinations each with value not less than 2, then such a sum may be replaced by a product. This procedure is carried out as far as possible. The process must end, since each such replacement increases the number of factors, and this cannot be larger than $r+s+t$. The only factors which cannot be replaced in this way are those listed in the conclusion of the lemma.

We return to the proof of Theorem 1. It is sufficient to assume that a combination satisfying the conclusion of Lemma 2 is given. Let g_1, g_{11}, g_{111}, g_2, g_{21}, g_0, and g_{01} be, respectively, the number of factors in the combination which are of the form 1, (1+1), (1+1+1), 2, (2+1), a_j, or (a_j+1) for $1\leqq j\leqq t$.

First, if $g_{01} \geqq 2$ and, after rearrangement of factors, $1 \leqq j < k \leqq t$, then a larger combination can be obtained by use of the inequality $(a_j+1)\cdot(a_k+1) = a_j \cdot a_k + a_j + a_k + 1 < a_j \cdot a_k + 3 \cdot a_k \leqq (1+1)\cdot a_j \cdot a_k$.

If, however, $g_{01} = 1$, then one of the inequalities $1\cdot(a_j+1) < (1+1)\cdot a_j$, $(1+1)\cdot(a_j+1) < (1+1+1)\cdot a_j$, $(1+1+1)\cdot(a_j+1) \leqq (1+1)^2 \cdot a_j$, $2\cdot(a_j+1) < (2+1)\cdot a_j$, $(2+1)\cdot(a_j+1) \leqq (1+1)\cdot 2\cdot a_j$ for $1 \leqq j \leqq t$ can be applied to give a combination of not smaller value with $g_{01} = 0$, unless both $r = 1$ and $s = 0$. (Note that r cannot be 0, since $g_{01} \leqq r$.)

If $g_{01} = 1$, $r = 1$, and $s = 0$, then use of the inequality $a_1 \cdot (a_j+1) \leqq (a_1+1)\cdot a_j$ with $1 < j \leqq t$ yields form (b).

If $g_{01} = 0$, $r = 1$, and $s = 0$, then either $t = 0$, yielding form (a), or $t \geqq 1$, in which case use of the inequality $1 \cdot a_1 < a_1 + 1$ yields a combination of larger value (which is then in form (b)).

The remaining cases then have $g_{01} = 0$ and either $r \neq 1$ or $s > 0$.

If, now, $g_1 \geqq 2$, then a combination of larger value is obtained by use of the inequality $1 \cdot 1 < 1 + 1$.

If $g_1 = 1$, then $r \geqq 2$ (since $g_1 \leqq r \neq 1$). Hence $g_1 < r$, so that at least one of the numbers g_{11}, g_{111}, g_{21} is positive (since $g_{01} = 0$). Hence use of one of the inequalities $1\cdot(1+1) < 1+1+1$, $1\cdot(1+1+1) < (1+1)^2$, $1\cdot(2+1) < (1+1)\cdot 2$ yields a combination of larger value.

Thus it is now necessary to consider only those cases in which $g_1 = g_{01} = 0$. But then $g_0 = t$, and a count of 1's and 2's shows that

$$2 \cdot g_{11} + 3 \cdot g_{111} + g_{21} = r, \tag{1}$$

$$g_2 + g_{21} = s. \tag{2}$$

Hence

$$2 \cdot g_{11} + 3 \cdot g_{111} = g_2 + r - s. \tag{3}$$

If, now, $r \leqq s$, then (3) yields the inequalities

$$g_{111} \leqq 2 \cdot g_{11} + 3 \cdot g_{111} \leqq g_2. \tag{4}$$

Sufficient use of the identity $(1+1+1)\cdot 2 = (1+1)\cdot(2+1)$ yields a combination with $g_{111} = 0$. Then (4) shows that the new combination has $2 \cdot g_{11} \leqq g_2$.

If $g_{11} > 0$, then $g_2 \geqq 2$, so that the inequality $(1+1)\cdot 2^2 < (2+1)^2$ can be applied to yield a combination with larger value. But if $g_{11} = 0$, then $g_{21} = r$ and $g_2 = s - r$, so that the combination is in form (c).

Therefore, it is now necessary to consider only the case $r > s$ (with the conditions $g_1 = g_{01} = 0$, $g_0 = t$, and either $r \neq 1$ or $s > 0$). Then one of the strict inequalities

$$\begin{gathered} (1+1)\cdot 2^2 < (2+1)^2, \\ (1+1)^2 \cdot 2 < (1+1+1)\cdot(2+1), \\ (1+1)^3 < (1+1+1)^2, \end{gathered} \tag{5}$$

yielding a combination of larger value, is applicable unless

(6) $$g_2 \geqq 2 \quad \text{and} \quad g_{11} = 0, \text{ or}$$

(7) $$g_2 = 1 \quad \text{and} \quad g_{11} \leqq 1, \text{ or}$$

(8) $$g_2 = 0 \quad \text{and} \quad g_{11} \leqq 2.$$

Now sufficient use of the identity $(1+1+1)\cdot 2=(1+1)\cdot(2+1)$ yields a combination with either

(9) $$g_{111} = 0 \quad \text{or} \quad g_2 = 0.$$

One of the inequalities (5) may now be applicable, yielding a combination with larger value; if (5) is not applicable, then (3), together with the condition $r>s$, yields

(10) $$2\cdot g_{11} + 3\cdot g_{111} > g_2.$$

If, now, $g_2 \geqq 2$, then $g_{11}=g_{111}=0$ (from (6) and (9)). But this contradicts (10). Hence $g_2 \leqq 1$.

If $g_2=1$, then $g_{11} \leqq 1$ and $g_{111}=0$ (from (7) and (9)). But this contradicts (10) unless $g_{11}=1$, in which case (1) and (2) imply that $g_{21}=r-2=s-1$, so that the combination is in form (d).

If $g_2=0$, then (2) implies that $g_{21}=s$, so that $3\cdot g_{111}=r-s-2\cdot g_{11}$ (from (1)). But then (8) implies that one of the following holds:

$$g_{11} = 0 \quad \text{and} \quad g_{111} = (r-s)/3, \text{ or}$$

$$g_{11} = 1 \quad \text{and} \quad g_{111} = (r-s-2)/3, \text{ or}$$

$$g_{11} = 2 \quad \text{and} \quad g_{111} = (r-s-4)/3.$$

Hence the combination is in form (f), (g), or (e), respectively (or in form (h)).

The proof of Theorem 1 is now complete, since it has been shown that every combination either is of the same value as one of the forms (a)–(g) or can be replaced by a combination of larger value. Since there are only finitely many combinations, the replacement process must end.

5. Extensions. In this section two additional operations, namely subtraction and negation, will be allowed. For example, the combination (-3) involves negation but not subtraction.

THEOREM 2. *With the same hypotheses as in Theorem* 1, *except that subtraction and negation, as well as addition and multiplication, are considered to be allowed operations, the same conclusions follow. In other words, subtraction and negation are never necessary.*

Proof. Let V be any combination in which addition, subtraction, multiplication, and negation may be used. By use of the associative, commutative, and distributive laws and the laws of signs, V is multiplied out without simplification. An expression in sum-product form (which need not be a combination) is

obtained. A new expression is formed by making all signs positive. This new expression has a value which is not smaller than the value of V. But this new expression is easily seen to be equal in value to the combination obtained from V by changing subtraction to addition and omitting negation. Hence V can be replaced by a combination not involving subtraction or negation and having a not smaller value.

It is conjectured that if division and reciprocation, as well as subtraction and negation, are also made allowed operations, then they are never necessary to achieve a maximizing combination.

6. Application. It is easy to construct combinations (allowing addition, subtraction, multiplication, and negation) using the digits 1, 2, 3, 4 once each, and having any integer value from 1 through 28, or the value 30, 32, or 36. By means of Theorem 2 it is possible to show that no other values are obtainable.

For example, it will be shown that 29 cannot be obtained as value of a combination; the proof for any other excluded value is similar. Now if 29 were obtainable using 1, 2, 3, 4, then it would be obtainable using ten 1's, since 2, 3, and 4 can be replaced by the proper sums of 1's in any combination.

Let $M(n)$ denote the largest number obtainable using exactly n 1's (and no other numbers). The table below lists a few convenient values of $M(n)$ and $M(10-n)$.

n	1	2	3	4	5	6	7	8	9	10
$M(n)$	1	2	3	4	6	9	12	18	27	36
$M(10-n)$	27	18	12	9	6	4	3	2	1	—

In evaluating a combination with value 29, the last ("outermost") operation performed must be an addition, subtraction, or multiplication of two *proper* subcombinations. It may be assumed that each of these proper subcombinations has nonnegative value; otherwise a different combination having these properties could be obtained by an obvious change of signs. Each of the subcombinations must involve not more than 9 1's. If subtraction is the last operation, then the minuend must have value at least 29 (since the subtraend is nonnegative). If multiplication is the last operation, then one of the factors must have value 29. But both of these are impossible, since $M(9)<29$. If, however, addition is the last operation, then 29 cannot be obtained, since $M(n)+M(10-n)<29$ if $n\leqq 9$. Hence 29 is unobtainable.

A later paper will contain results concerning constructions using numbers which need not be integers.

THE INTEGRABILITY OF CERTAIN FUNCTIONS AND RELATED SUMMABILITY METHODS II

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I return to my previous paper (this MONTHLY, vol. 66, 1959, pp. 361–375) to add the following three remarks which are independent of each other.

Remark 1. Professor A. S. Besicovitch communicated to me the following beautiful direct proof of the property (1.5) of the fractions (1.4): We are to prove that *to any* $\delta>0$ *there corresponds an* N *such that if* $n>N$ *then one of the* ϕ *fractions* (1.4) *falls in any subinterval* I_δ *of* $[0,1]$, *of length* δ. To prove this, denote by $p_1, \cdots, p_k$ all distinct prime factors of n. Given $\delta>0$ there corresponds an N such that if $n>N$ then at least one of the inequalities

$$\text{(i)} \qquad (p_1 \cdots p_k)/n < \delta, \qquad \text{(ii)} \qquad 2/p_k < \delta$$

holds. If (i) holds then any I_δ contains an irreducible fraction of the form $(qp_1 \cdots p_k+1)/n$.

Let now (ii) hold. Any I_δ contains at least $[n\delta]$ fractions of the form m/n; but $n\delta > 2n/p_k \geqq 2p_1 \cdots p_{k-1}$. Hence an I_δ contains at least $2p_1 \cdots p_{k-1}$ consecutive fractions m/n. However, any $2p_1 \cdots p_{k-1}$ consecutive integers m contain a pair of numbers of the form

$$q \cdot p_1 \cdots p_{k-1} + 1, \qquad (q+1)p_1 \cdots p_{k-1} + 1$$

and at least one of these, call it m, is prime to n. But then m/n is in I_δ and the proof is complete.

Remark 2. In his paper *On the limits of Riemann approximating sums*, Quart. J. Math., Oxford Ser., vol. 18, 1947, pp. 124–127, P. Hartman establishes an interesting result for vector-valued functions which, stated in two dimensions, is as follows: *Let* $f(x)$ *be real- or complex-valued and bounded in* $[0,1]$. *An expression of the form*

$$S = \sum_{\nu=1}^{n} f(\xi_\nu)(x_\nu - x_{\nu-1}), \qquad (x_0 = 0 < x_1 < \cdots < x_n = 1, \qquad x_{\nu-1} \leqq \xi_\nu \leqq x_\nu),$$

is called, as usual, a Riemann sum. Let $\{S_n\}$ *be an infinite sequence of R-sums with the following two properties*: (i) *The maximal subinterval of* S_n *converges to zero as* $n\to\infty$, (ii) *The limit* $\lim_{n\to\infty} S_n = \sigma$ *exists. Let* $R=\{\sigma\}$ *denote the set of all such limits. Then* R *is a bounded and closed convex domain in the complex plane.*

It is now clear that $f(x)$ is Riemann integrable if and only if the convex set R reduces to a point, *the* value of the integral.

Let us now take a second look at our special function (2.1), where λ and the elements of the sequence $\{\gamma_n\}$ are now real- or complex-valued. The following theorem generalizes our conditions (2.2) for the R-integrability of $f(x)$:

THEOREM 6. *The Riemann-Hartman set R for the function $f(x)$, defined by* (2.1), *is described by*

$$R = K(\lambda \cup \{\lim \gamma_n\}),$$

where the right side is the closed convex hull of the union of the point λ with the set of limit points of the sequence $\{\gamma_n\}$.

Proof. Let the set $\Gamma = \lambda \cup \{\lim \gamma_n\}$ be contained in the open half-plane $H: u \cos\theta + v \sin\theta - h < 0$ of the complex plane $u+iv$. Now Riemann sums over the range $[0, 1]$ are of the nature of centroids; if $\{S_n\}$ is a sequence of R-sums satisfying Hartman's conditions (i) and (ii), then it is clear from (2.1) that also the point $S_n = u_n + iv_n$ will be in the half-plane H, provided that n is sufficiently large. This proves that $R \subseteq K(\Gamma)$.

In order to prove the opposite inclusion, let $\sigma \in K(\Gamma)$ and let us show that $\sigma \in R$. Since Γ is a closed set, σ may be represented as $\sigma = \alpha a + \beta b + \gamma c$, where $a, b, c \in \Gamma$, $\alpha, \beta, \gamma \geqq 0$, $\alpha+\beta+\gamma = 1$. To fix the ideas let α, β, γ, be all positive. Given $\epsilon > 0$ we can evidently construct for our function $f(x)$ three R-sums S_a, S_b, S_c, all three corresponding to the same division $x_0 = 0 < x_1 < \cdots < x_n = 1$, differing only in the ξ's, and such that

$$|S_a - a| < \epsilon, \quad |S_b - b| < \epsilon, \quad |S_c - c| < \epsilon, \quad x_\nu - x_{\nu-1} < \epsilon, \quad (\nu = 1, \cdots, n).$$

Let now i and j be integers defined by the inequalities

$$x_{i-1} \leqq \alpha < x_i, \qquad x_{j-1} \leqq \alpha + \beta < x_j.$$

If we now form a new R-sum S^* obtained by taking the ξ_ν of S_a if $\nu = 1, \cdots, i$, the ξ_ν of S_b if $\nu = i+1, \cdots, j$, and finally those of S_c if $\nu = j+1, \cdots, n$, then evidently S^* will be close to $\sigma = \alpha a + \beta b + \gamma c$ and in fact as close as we wish provided that ϵ is sufficiently small. This shows that $\sigma \in R$ and completes the proof.

Theorem 6 immediately yields the following *converse* of Hartman's theorem:

THEOREM 7. *Any given bounded and closed convex domain D in the complex plane may serve as the Riemann-Hartman domain R of an appropriate bounded function $f(x)$ in $[0, 1]$, in fact functions of the type* (2.1) *suffice.*

Proof. Indeed, select at will $\lambda \in D$ and also the sequence $\{\gamma_n\}$ so as to be everywhere dense on the boundary of D. Theorem 6 now shows that for the function (2.1) we have $R = D$.

Remark 3. I was kindly informed by Professor G. Szegö that the remark of Section 6 was already made long ago by the then very young Hungarian mathematician, F. Lukács, and published by him in Math. Annalen, vol. 70, 1911, pp. 561–562.

A SPECIAL VANDERMONDIAN DETERMINANT

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The primary problem under consideration here is that of the evaluation of a particular determinant whose general term is of the form

$$a_{ij} = x_i^j, \qquad i, j = 0, 1, \cdots, m,$$

where $x_i = \cos\theta_i$, $\theta_i = \pi i/m$, $m \geqq 1$, and a_{i0} is taken to be one if x_i equals zero. This particular case might arise if one attempted to solve the problem of passing a polynomial of degree m through $m+1$ points, whose (Chebyshev) abscissas are as given by the above cosine formula, by the use of determinants. The resulting coefficient determinant is the one that interests us here.

We will show that it has a value D such that

$$D^2 = m^{m+1}/2^{m^2-3}.$$

Thus, D is relatively small for moderately large m. For example, for $m = 20$, we find that D is approximately $8(10)^{-47}$, and hence a reasonable amount of care would have to be taken to obtain a numerical solution of any significance.

In the course of the proof, various identities involving binomial coefficients arise which are of perhaps more interest than the basic evaluation referred to initially. For example, in the first part of the proof, we require the identities indicated by the numbers appearing on and above the principal diagonal in the product of the following infinite matrices:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \\ & 1 & 2^{-1}\binom{1}{0} & 2^{-3}\binom{3}{1} & 2^{-5}\binom{5}{2} & 2^{-7}\binom{7}{3} & \cdots \\ & & 2^{-1}\binom{1}{0} & 2^{-3}\binom{3}{1} & 2^{-5}\binom{5}{2} & 2^{-7}\binom{7}{3} & \cdots \\ & & & 2^{-3}\binom{3}{1} & 2^{-5}\binom{5}{2} & 2^{-7}\binom{7}{3} & \cdots \\ & & & & 2^{-5}\binom{5}{2} & 2^{-7}\binom{7}{3} & \cdots \\ & & & & & \cdots & \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & \cdots \\ & 2^2 & -2^2\binom{3}{1} & 2^2\binom{4}{2} & -2^2\binom{5}{3} & \cdots \\ & & 2^4 & -2^4\binom{5}{1} & 2^4\binom{6}{2} & \cdots \\ & & & 2^6 & -2^6\binom{7}{1} & \cdots \\ & & & & 2^8 & \cdots \\ & & & & & \cdots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 5 & 7 & 9\cdots \\ & 1 & 1 & 1 & 1\cdots \\ & & 1 & 1 & 1\cdots \\ & & & 1 & 1\cdots \\ & & & & 1\cdots \end{pmatrix}.$$

Such identities are established through the use of formal manipulations with power series.

Since it seems that these identities, as well as the method of proof, are interesting, the procedure used will be outlined. However, as most of the elements of the proof involve mathematical induction, and are rather long and tedious, details will be omitted.

We first obtain the determinant for D^2 by multiplying the ith column of D by the jth column of D. This gives a determinant whose general term is of the form

$$b_{ij} = s_{i+j} \qquad i, j = 0, 1, \cdots, m,$$

where $s_{i+j} = x_0^{i+j} + x_1^{i+j} + \cdots + x_m^{i+j}$.

Since the x's are symmetric with respect to the origin, $s_{i+j} = 0$ for $i+j$ odd. Hence

$$D^2 = \begin{vmatrix} s_0 & 0 & s_2 & 0 & s_4 \cdots \\ 0 & s_2 & 0 & s_4 & 0 \cdots \\ s_2 & 0 & s_4 & 0 & s_6 \cdots \\ 0 & s_4 & 0 & s_6 & 0 \cdots \\ s_4 & 0 & s_6 & 0 & s_8 \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \cdots \end{vmatrix},$$

or, upon rearranging the columns from the order $1, 3, 5, \cdots, 2, 4, \cdots$, to the order $1, 2, 3, 4, 5, \cdots$, followed by a corresponding row rearrangement, we obtain

$$D^2 = -\begin{vmatrix} s_0 & s_2 & s_4 \cdots & 0 & 0 \\ 0 & 0 & 0 \cdots & s_2 & s_4 \\ s_2 & s_4 & s_6 \cdots & 0 & 0 \\ 0 & 0 & 0 \cdots & s_4 & s_6 \\ s_4 & s_6 & s_8 \cdots & 0 & 0 \\ \cdot & \cdot & \cdot \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot \cdots & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} s_0 & s_2 & s_4 \cdots & 0 & 0 \cdots \\ s_2 & s_4 & s_6 \cdots & 0 & 0 \cdots \\ s_4 & s_6 & s_8 \cdots & 0 & 0 \cdots \\ \cdot & \cdot & \cdot \cdots & \cdot & \cdot \cdots \\ 0 & 0 & 0 \cdots & s_2 & s_4 \cdots \\ 0 & 0 & 0 \cdots & s_4 & s_6 \cdots \\ \cdot & \cdot & \cdot \cdots & \cdot & \cdot \cdots \end{vmatrix} = \begin{vmatrix} s_0 & s_2 & s_4 \cdots \\ s_2 & s_4 & s_6 \cdots \\ s_4 & s_6 & s_8 \cdots \\ \cdot & \cdot & \cdot \cdots \end{vmatrix} \cdot \begin{vmatrix} s_2 & s_4 \cdots \\ s_4 & s_6 \cdots \\ \cdot & \cdot \cdots \\ \cdot & \cdot \cdots \end{vmatrix}.$$

To obtain a simple expression for s_{i+j}, we establish next that

$$2^{2n-1} \cos^{2n} x = a_0 + a_2 \cos 2x + \cdots + a_{2n} \cos 2nx, \tag{1}$$

where $a_i = a_i(n)$ and

$$a_0 = \binom{2n-1}{n-1}, \qquad a_{2n} = 1,$$

$$a_{2i} = \binom{2n-1}{n-1-i} + \binom{2n-1}{n-1+i}, \qquad i = 1, \cdots, n-1.$$

Using (1) and the standard identity

$$\cos 0y + \cos 1y + \cos 2y + \cdots + \cos my = \tfrac{1}{2}[1 + \sin (m + \tfrac{1}{2})y/\sin \tfrac{1}{2}y]$$

with $y = 2\pi/m, 4\pi/m, \cdots, 2(m-1)\pi/m$, we find that

$$s_0 = m + 1,$$

$$s_{2j} = 1 + \binom{2j-1}{j-1} m/2^{2j-1}, \qquad j = 1, \cdots, m-1,$$

$$s_{2m} = 1 + \left[\binom{2m-1}{m-1} + 1\right] m/2^{2m-1}.$$

Noting the product expression for D^2 and the form of the elements involved therein, we consider first the sequence of determinants

$$|t_0|, \quad \begin{vmatrix} t_0 & t_2 \\ t_2 & t_4 \end{vmatrix}, \quad \begin{vmatrix} t_0 & t_2 & t_4 \\ t_2 & t_4 & t_6 \\ t_4 & t_6 & t_8 \end{vmatrix}, \cdots,$$

where $t_0 = 1+x$, $t_{2j} = 1 + \binom{2j-1}{j-1} x/2^{2j-1}$, $j = 1, 2, \cdots$. Since the pth determinant in this sequence has elements which reduce to 1 when $x = 0$, its expression has a factor x^{p-1}.

We next find a set of constant multipliers $a_i = a_i(p)$ such that

$$(2) \qquad a_0 t_q + a_2 t_{q+2} + \cdots + a_{2p-2} t_{q+2p-2} = x + 2p - 1, \quad q = 0, 2, \cdots, 2p-2,$$

proving that $x+2p-1$ is also a factor of the expression just referred to.

For each p, we show that the $a_i(p)$ are the first p numbers in the pth column of the infinite matrix appearing below:

$$\begin{pmatrix} (1+z)^{-1} \\ (4z)(1+z)^{-3} \\ (4z)^2(1+z)^{-5} \\ (4z)^3(1+z)^{-7} \\ \cdots\cdots \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 & \cdots \\ 0 & 4 & -\binom{3}{1}4 & \binom{4}{2}4 & \cdots \\ 0 & 0 & 4^2 & -\binom{5}{1}4^2 & \cdots \\ 0 & 0 & 0 & 4^3 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix} \begin{pmatrix} 1 \\ z \\ z^2 \\ z^3 \\ \cdot \end{pmatrix}.$$

For it follows from this identity and the relation

$$(1-u)^{-1/2} = 1 + \binom{1}{0} u/2 + \binom{3}{1} u^2/2^3 + \binom{5}{2} u^3/2^5 + \cdots,$$

that

$$(1+z)^{-1} + \binom{1}{0}(4z)(1+z)^{-3}/2 + \binom{3}{1}(4z)^2(1+z)^{-5}/2^3 + \cdots$$

$$= [1 - (4z)(1+z)^{-2}]^{-1/2}/(1+z) = 1 + z + z^2 + \cdots$$

$$= 1 + \left[-1\cdot 1 + 4\cdot \binom{1}{0}\Big/ 2\right]z$$

$$+ \left[1\cdot 1 - \binom{3}{1}4\cdot \binom{1}{0}\Big/ 2 + 4^2\cdot \binom{3}{1}\Big/ 2^3\right]z^2 + \cdots .$$

Comparing coefficients, we find that

$$a_0(1)t_0' = 1, \qquad a_0(2)t_0' + a_2(2)t_2' = 1,$$
$$a_0(3)t_0' + a_2(3)t_2' + a_4(3)t_4' = 1, \cdots,$$

where the prime denotes differentiation.

Similarly, using the same expansion for $(1-u)^{-1/2}-1$, we have

$$\binom{1}{0}(1+z)^{-1}/2 + \binom{3}{1}(4z)(1+z)^{-3}/2^3 + \cdots$$

$$= [(1+z)/(4z)][\{1 - (4z)/(1+z^2)\}^{-1/2} - 1] = \tfrac{1}{2} + z + z^2 + \cdots$$

$$= \binom{1}{0}\Big/ 2 + \left[-1\cdot \binom{1}{0}\Big/ 2 + 4\cdot \binom{3}{1}\Big/ 2^3\right]z + \cdots ;$$

or, beginning with the second coefficient,

$$a_0(2)t_2' + a_2(2)t_4' = 1, \qquad a_0(3)t_2' + a_2(3)t_4' + a_4(3)t_6' = 1, \cdots .$$

Continuing this process, using the expansion of $(1-u)^{-1/2}$ with the first q terms transposed as indicated above, we are led to an expression in which all coefficients after the first q are ones. This, in turn, leads to identities involving $a(q+1)$, $a(q+2)$, $\cdots$. To conclude the present part of the argument, we have

$$(1+z)^{-1} + (4z)(1+z)^{-3} + \cdots$$

$$= (1+z)^{-1}/[1 - (4z)(1+z)^{-2}] = \sum_{j=1}^{\infty}(2j-1)z^{j-1}$$

$$= 1 + (-1\cdot 1 + 4\cdot 1)z + \cdots ,$$

implying

$$a_0(1)t_0(0) = 1, \qquad a_0(2)t_0(0) + a_2(2)t_2(0) = 3,$$
$$a_0(3)t_0(0) + a_2(3)t_2(0) + a_4(3)t_4(0) = 5, \cdots .$$

Since $t_{2j}(0) = 1$, these relations are sufficient to prove (2).

Thus, the pth determinant in our sequence is of the form $C_p x^{p-1}(x+2p-1)$, since it is obviously a pth degree polynomial in x. To determine C_p, each determinant can be evaluated in terms of the one preceding it. For example, consider the third determinant. Multiplying the rows by the appropriate a's, and adding

to the last row, we obtain

$$C_3x^2(x+5)=\begin{vmatrix} t_0 & t_2 & t_4 \\ t_2 & t_4 & t_6 \\ t_4 & t_6 & t_8 \end{vmatrix}=(1/4^2)\begin{vmatrix} t_0 & t_2 & t_4 \\ t_2 & t_4 & t_6 \\ x+5 & x+5 & x+5 \end{vmatrix},$$

or

$$C_3x^2=(1/4^2)\begin{vmatrix} t_0 & t_2 & t_4 \\ t_2 & t_4 & t_6 \\ 1 & 1 & 1 \end{vmatrix}.$$

Repeating the process using the columns, we find that

$$C_3x^2=(1/4^2)^2\begin{vmatrix} t_0 & t_2 & x+5 \\ t_2 & t_4 & x+5 \\ 1 & 1 & 5 \end{vmatrix}$$

or that $C_3(-5)^2=(5)C_2(-5)^1(-5+3)/4^4$. For the general term, proceeding similarly, we obtain

$$C_p=1/2^a, \quad \text{where} \quad \begin{cases} a=0, & p=1, \\ a=\binom{2p-1}{2}, & p>1. \end{cases}$$

The sequence

$$|t_2|, \quad \begin{vmatrix} t_2 & t_4 \\ t_4 & t_6 \end{vmatrix}, \cdots$$

is treated similarly. Multipliers, leading to a factor $x+2p$, appear in the matrix below:

$$\begin{Bmatrix} 2(1+z)^{-2} \\ 2^2z(1+z)^{-4} \\ 2^4z^2(1+z)^{-6} \\ 2^7z^3(1+z)^{-8} \\ \cdot \end{Bmatrix}=\begin{Bmatrix} 1 & -2\cdot 2 & 2.3 & -2.4 & \cdots \\ 0 & 2^3 & -2^3\binom{4}{1} & 2^3\binom{5}{2}\cdots & \\ 0 & 0 & 2^5 & -2^5\binom{6}{1}\cdots & \\ 0 & 0 & 0 & 2^7 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{Bmatrix}\begin{Bmatrix} 1 \\ z \\ z^2 \\ z^3 \\ \cdot \end{Bmatrix}.$$

Following an identical procedure, using the series for $(1-u)^{-1/2}$, we find that the pth determinant is equal to $D_px^{p-1}(x+2p)$, where

$$D_p=1/2^b, \qquad b=\binom{2p}{2}.$$

Combining these results, we have

$$T_{2p} = \begin{vmatrix} t_0 & 0 & t_2 & 0 \cdots \\ 0 & t_2 & 0 & t_4 \cdots \\ t_2 & 0 & t_4 & 0 \cdots \\ 0 & t_4 & 0 & t_6 \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix} = x^{p-1}(x+p)(x+p+1)/2^e,$$

$p \geqq 1$, for a determinant of order $p+1$, where $e = p^2$.

Returning now to D^2 for general m and letting $x = m$, we find that D^2 can be decomposed, since all t's and s's are identical except that $s_{2m} = t_{2m} + x/2^{2m-1}$, into a sum of two T-determinants. Thus, assuming that $t_k = 0$ for k odd,

$$\begin{aligned} D^2 &= \begin{vmatrix} t_0 \cdots t_m \\ \cdot \quad \cdots \quad \cdot \\ t_m \cdots t_{2m} + x/2^{2m+1} \end{vmatrix} \\ &= \begin{vmatrix} t_0 \cdots t_m \\ \cdot \quad \cdots \quad \cdot \\ t_m \cdots t_{2m} \end{vmatrix} + \begin{vmatrix} t_0 \cdots t_{m-1} & 0 \\ \cdot \quad \cdots \quad \cdot & \cdot \\ t_m \cdots t_{2m-1} & x/2^{2m-1} \end{vmatrix} \\ &= T_{2m} + (x/2^{2m-1})T_{2m-2} = x^{m-1}(x+m)^2/2^{m^2-1} \\ &= m^{m+1}/2^{m^2-3}. \end{aligned}$$

For the Chebyshev subdivision $\theta_i = (2i-1)\pi/(2m)$, $i = 1, \cdots, m$, using

$$\cos y + \cos 3y + \cdots + \cos(2m-1)y = (\sin 2my)/(2 \sin y)$$

with $y = \pi/m, 2\pi/m, \cdots, (m-1)\pi/m$, we can show that

$$s_0 = m, \qquad s_2 = \binom{1}{0} m/2, \cdots, s_{2m-2} = \binom{2m-3}{m-2} m/2^{2m-3}.$$

If we now divide each term in T_{2p} by x and then let $x \to \infty$, we obtain, associating $p+1$ with m,

$$\begin{vmatrix} 1 & 0 & \binom{1}{0}/2 \cdots \\ 0 & \binom{1}{0}/2 & 0 \quad \cdots \\ \binom{1}{0}/2 & 0 & \binom{3}{1}/2 \cdots \\ \cdot & \cdot & \cdot \quad \cdots \end{vmatrix} = 1/2^{(m-1)^2}.$$

Comparing these numbers with the formulas for the s's, we conclude that, for this case, $D^2 = m^m/2^{(m-1)^2}$.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, Kent State University

The following results of the nineteenth William Lowell Putnam Mathematical Competition held on November 22, 1958, have been determined in accordance with the constitution of the competition. This competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were Richard M. Dudley, Robert C. Hartshorne, and Stephen Lichtenbaum; to each of these a prize of fifty dollars is awarded.

The second prize, four hundred dollars, is awarded to the Department of Mathematics of the University of Toronto, Toronto, Ontario. The members of the team were David Brillinger, John Hewett, and Joseph Lipman; to each of these a prize of forty dollars is awarded.

The third prize, three hundred dollars, is awarded to the Department of Mathematics of the California Institute of Technology, Pasadena, California. The members of the team were Alfred W. Hales, Robert Jewett, and Thomas Morton; to each of these a prize of thirty dollars is awarded.

The fourth prize, two hundred dollars, is awarded to the Department of Mathematics of Cornell University, Ithaca, New York. The members of the team were Jim Bennett, Jack Newman, and Barbara Osofsky; to each of these a prize of twenty dollars is awarded.

The fifth prize, one hundred dollars, is awarded to the Department of Mathematics of the Polytechnic Institute or Brooklyn, Brooklyn, New York. The members of the team were George Glauberman, Martin Isaacs, and Donald Passman; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, are Alfred W. Hales, California Institute of Technology; Robert C. Hartshorne, Harvard University; John Hewett, University of Toronto; Joseph Lipman, University of Toronto; and Alan Gaisford Waterman, San Diego State College. Each of these will receive a prize of seventy-five dollars.

The five succeeding persons ranking highest in the examination, named in alphabetical order, are Stephen Adler, Harvard University; Richard M. Dudley, Harvard University; Samuel Klein, College of the City of New York; Stephen Lichtenbaum, Harvard University; and Gerald Stoller, Polytechnic Institute of Brooklyn. Each of these will receive a prize of thirty-five dollars.

The following teams, named in alphabetical order, won honorable mention: Case Institute of Technology, Cleveland, Ohio, the members of the team being Donald E. Knuth, William C. Lynch, and George W. Petznick; Columbia

College, New York City, the members of the team being Joseph D'Atri, Benjamin Bennett, and Paul B. Kantor; Massachusetts Institute of Technology, Cambridge, Massachusetts, the members of the team being David Carey, John Drinks, and Steven Scheinberg; University of California, Berkeley, California, the members of the team being Stefan A. Burr, Philip Read, and Floris Y. Tsang; and Yale University, New Haven, Connecticut, the members of the team being R. W. Beals, David Book, and Lee D. Goldberg.

Eleven individuals were given honorable mention. Their names, alphabetically arranged, are: R. W. Beals, Yale University; David Handel, California Institute of Technology; Robert Jewett, California Institute of Technology; John F. Kennison, Queens College (New York); Donald E. Knuth, Case Institute of Technology; Eugene Luks, College of the City of New York; Donald Olivier, Carleton College; Barbara Osofsky, Cornell University; Donald Passman, Polytechnic Institute of Brooklyn; Thomas Morton, California Institute of Technology; and Steven Scheinberg, Massachusetts Institute of Technology.

A total of six hundred and twenty-nine contestants from one hundred and nineteen institutions entered the competition this year. Of this number, one hundred and twenty-three individuals and four institutions were unable to compete, due to various reasons. Therefore, a total of five hundred and six undergraduates from one hundred and fifteen institutions actually took part in the competition.

The following is a list, alphabetically arranged, of all colleges and universities which entered teams in the competitition:

Agricultural and Mechanical College of Texas, Agricultural and Technical College of North Carolina, American University, Anna Maria College, Arizona State University, Atlanta University System, Belhaven College, Brooklyn College, Brown University, California Institute of Technology, Carleton College, Case Institute of Technology, Centenary College, Central Michigan College, College of Saint Catherine, College of Saint Thomas, College of the City of New York, College of the Holy Cross, Columbia College, Cornell University, Dartmouth College, Drew University, Eastern Baptist College, Eastern New Mexico University, Georgetown University, Georgia Institute of Technology, Harvard University, Humboldt State College, Iowa State College, Jamestown College, Kenyon College, Knox College, Lafayette College, Lebanon Valley College, Mankato State College, Massachusetts Institute of Technology, McGill University, Memphis State University, Mississippi State University, New Mexico College of Agriculture and Mechanics, Norwich University, Oberlin College, Oklahoma State University, Oregon State College, Pasadena College, Polytechnic Institute of Brooklyn, Pomona College, Princeton University, Purdue University, Queens College (Flushing, N. Y.), Queen's University (Kingston, Ontario), Radcliffe College, Reed College, Rutgers University, Saint Francis Xavier University, Saint Martins College, San Jose State College, Santa Clara University, Siena College, South Dakota School of Mines, Southwestern at Memphis, Stanford University, State College of Washington, Stevens Institute of Technology, Swarthmore College, The Cardinal Stritch College, The Ohio State University, The Rice Institute, Tufts University, United States Naval Aacdemy, University of Alberta, University of Arizona, University of British Columbia, University of California (Berkeley), University of California (Davis), University of California (Los Angeles), University of Colorado, University of Idaho, University of Illinois, University of Kansas, University of Manitoba, University of Michigan, University of Minnesota, University of Notre Dame, University of Oregon, University of Rochester, University of Santa Clara, University of the South, University of Texas, University of

Toronto, University of Washington, University of Western Ontario, University of Wisconsin, Wake Forest College, Washington University (St. Louis), Wayne State University, Wesleyan University, Yale University, and Yeshiva College.

The following colleges and universities, alphabetically arranged, entered individual contestants only:

Aurora College, Brandeis University, Colorado College, Duquesne University, Haverford College, Kent State University, Lincoln Memorial University, Northwestern University, Rocky Mountain College, Sacramento State College, Saint Mary-of-the-Wasatch, Saint Olaf College, San Diego State College, University of Chicago, University of Cincinnati, University of Florida, University of Houston, University of Kentucky, University of New Hampshire, University of Ottawa, and Wofford College.

The individual ranks of contestants (except for the relative ranks of the first five) may be obtained by any department of mathematics for the purpose of selecting graduate students.

The problems given to those participating in the competition, together with a write-up of the solutions to the problems, will appear in a later issue of the MONTHLY.

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, University of California, Berkeley 4, California.

EXPONENTIATION OF PERMUTATION GROUPS*

FRANK HARARY, University of Michigan and Princeton University

1. In order to formulate a problem in the enumeration of linear graphs [1], we have defined the "exponentiation" $\mathfrak{B}^{\mathfrak{A}}$ and the "cartesian product" $\mathfrak{A}\times\mathfrak{B}$ of two permutation groups $\mathfrak{A}$ and $\mathfrak{B}$. Pólya [2] introduced the "Gruppenkranz" $\mathfrak{A}[\mathfrak{B}]$, or the "composition." The "direct sum" $\mathfrak{A}+\mathfrak{B}$, often also called the "direct product," is classical. Our object is to set forth some of the rules which govern the interaction of these operations with each other.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be permutation groups with disjoint object sets X and Y, degrees a and b, and orders m and n, respectively. We say that $\mathfrak{A}$ and $\mathfrak{B}$ are *abstractly isomorphic* and write $\mathfrak{A}\cong\mathfrak{B}$ when they are isomorphic regarded as abstract groups. They are *permutationally equivalent* (or *permutationally isomorphic*) if they are abstractly isomorphic and there is a 1-1 correspondence $f: X\leftrightarrow Y$ such that if θ is the abstract isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, then for all $x\in X$, $\alpha\in\mathfrak{A}$, we have $f(\alpha x)=(\theta\alpha)f(x)$. In this case we write $\mathfrak{A}=\mathfrak{B}$. Thus in particular $a=b$ whenever $\mathfrak{A}=\mathfrak{B}$.

* This work was supported by a grant from the Office of Naval Research.

The *direct sum* $\mathfrak{A}+\mathfrak{B}$ has $X \cup Y$ as its object set and its elements are all the permutation products $\alpha\beta$ obtained by juxtaposition of the permutations in $\mathfrak{A}$ with those in $\mathfrak{B}$.

The *cartesian product* $\mathfrak{A}\times\mathfrak{B}$ has $X \times Y$ as its object set and its elements (α, β) act on its objects (x, y) in accordance with the equation $(\alpha, \beta)(x, y) = (\alpha x, \beta y)$. Thus $\mathfrak{A}+\mathfrak{B} \cong \mathfrak{A}\times\mathfrak{B}$ but $\mathfrak{A}+\mathfrak{B} \neq \mathfrak{A}\times\mathfrak{B}$.

The *composition* $\mathfrak{A}[\mathfrak{B}]$ also has $X \times Y$ as its object set, but it is more convenient to regard the objects as the entries in an a by b matrix (x_{ij}). The elements of $\mathfrak{A}[\mathfrak{B}]$ are constructed by first permuting the matrix rows by an element of $\mathfrak{A}$ and then permuting the column indices in each of the a rows separately by an element of $\mathfrak{B}$, repetitions permitted.

The *exponentiation* $\mathfrak{B}^{\mathfrak{A}}$ has as object set Y^X, the collection of all functions from X into Y. The elements of $\mathfrak{B}^{\mathfrak{A}}$ are constructed by first permuting the domain X using a permutation in $\mathfrak{A}$ and then permuting the image objects for each domain object by elements of $\mathfrak{B}$, not necessarily distinct. It follows at once that $\mathfrak{B}^{\mathfrak{A}} \cong \mathfrak{A}[\mathfrak{B}]$ but in general $\mathfrak{B}^{\mathfrak{A}} \neq \mathfrak{A}[\mathfrak{B}]$.

More precisely, let $X = \{x_1, \cdots, x_a\}$, f be any function from X into Y, $\alpha \in \mathfrak{A}$, and $\beta_1, \cdots, \beta_a \in \mathfrak{B}$ (repetitions permitted). Then the following permutation γ of the exponentiation group $\mathfrak{B}^{\mathfrak{A}}$ is induced: $\gamma f = f^*$, where $f^*(x_i) = \beta_i f(\alpha x_i)$.

These definitions are summarized in Table 1.

TABLE 1

			Sum	Product	Composition	Exponentiation
Group	$\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{A}+\mathfrak{B}$	$\mathfrak{A}\times\mathfrak{B}$	$\mathfrak{A}[\mathfrak{B}]$	$\mathfrak{B}^{\mathfrak{A}}$
Object Set	X	Y	$X \cup Y$	$X \times Y$	$X \times Y$	Y^X
Degree	a	b	$a+b$	ab	ab	b^a
Order	m	n	mn	mn	mn^a	mn^a

2. We now find some permutational equivalences between certain combinations of these operations. Let $\mathfrak{S}_n$ be the symmetric group of degree n. It is convenient to consider not only $\mathfrak{S}_1$, the group of degree 1 and order 1, but also $\mathfrak{S}_0$, the group of degree 0 and order 1. Let $\mathcal{P}$ be the collection of all permutation groups. We omit the proof of the following theorem, which is immediate.

THEOREM 1. (a) *$\mathcal{P}$ is closed under sum, product, composition, and exponentiation.*

(b) *For all permutation groups $\mathfrak{A}$, the following identity rules hold:*

$$\mathfrak{A} + \mathfrak{S}_0 = \mathfrak{A}, \qquad \mathfrak{A} \times \mathfrak{S}_1 = \mathfrak{A}, \tag{1}$$
$$\mathfrak{A}[\mathfrak{S}_1] = \mathfrak{A}, \qquad \mathfrak{S}_1[\mathfrak{A}] = \mathfrak{A}, \qquad \mathfrak{A}^{\mathfrak{S}_1} = \mathfrak{A}.$$

(c) *The following commutativity laws hold:*

$$\mathfrak{A} + \mathfrak{B} = \mathfrak{B} + \mathfrak{A}, \qquad \mathfrak{A} \times \mathfrak{B} = \mathfrak{B} \times \mathfrak{A}. \tag{2}$$

(d) *The following associativity laws hold*:

$$\mathfrak{A}+(\mathfrak{B}+\mathfrak{C})=(\mathfrak{A}+\mathfrak{B})+\mathfrak{C}, \qquad \mathfrak{A}\times(\mathfrak{B}\times\mathfrak{C})=(\mathfrak{A}\times\mathfrak{B})\times\mathfrak{C}, \tag{3}$$
$$\mathfrak{A}[\mathfrak{B}[\mathfrak{C}]]=(\mathfrak{A}[\mathfrak{B}])[\mathfrak{C}].$$

We note, using Table 1, *that the orders of* $\mathfrak{A}[\mathfrak{B}]$ *and* $\mathfrak{B}[\mathfrak{A}]$ *are different so that* $\mathfrak{A}[\mathfrak{B}]\not\simeq\mathfrak{B}[\mathfrak{A}]$.

THEOREM 2. *Among the operations of sum, product, and composition, the only distributive law which holds is*:

$$(\mathfrak{A}+\mathfrak{B})[\mathfrak{C}]=\mathfrak{A}[\mathfrak{C}]+\mathfrak{B}[\mathfrak{C}]. \tag{4}$$

Proof. The fact that none of the other possible combinations yields permutational equivalence follows readily from Table 1 by either an order argument or a degree argument. For example, if c is the degree of $\mathfrak{C}$, then the degree of $(\mathfrak{A}\times\mathfrak{B})[\mathfrak{C}]$ is abc while the degree of $\mathfrak{A}[\mathfrak{C}]\times\mathfrak{B}[\mathfrak{C}]$ is abc^2.

To prove the assertion, consider the partitioned matrix:

$$M=\begin{bmatrix}M_1\\ M_2\end{bmatrix},$$

where

$$M_1=\begin{bmatrix}x_{11}&\cdots&x_{1c}\\ \vdots&&\vdots\\ x_{a1}&\cdots&x_{ac}\end{bmatrix}, \qquad M_2=\begin{bmatrix}y_{11}&\cdots&y_{1c}\\ \vdots&&\vdots\\ y_{b1}&\cdots&y_{bc}\end{bmatrix}.$$

Not only are the orders and the degrees of the two sides of (4) equal, but the above matrix M gives the 1-1 correspondence between their object sets under which they are permutationally equivalent. For the permutations in $\mathfrak{A}[\mathfrak{C}]$ act on the entries of submatrix M_1 while those in $\mathfrak{B}[\mathfrak{C}]$ act on M_2, and these transformations are independent.

We next turn to combinations of exponentiation with the other operations, and find the following two rules which are believed to be the only ones.

THEOREM 3. *The following two rules involving exponentiation hold*:

$$\mathfrak{A}^{\mathfrak{B}+\mathfrak{C}}=\mathfrak{A}^{\mathfrak{B}}\times\mathfrak{A}^{\mathfrak{C}}, \tag{5}$$

$$(\mathfrak{A}^{\mathfrak{B}})^{\mathfrak{C}}=\mathfrak{A}^{[\mathfrak{C}\mathfrak{B}]}. \tag{6}$$

It is immediately seen that if t is the order of $\mathfrak{C}$, then

$$ord\ \mathfrak{A}^{\mathfrak{B}+\mathfrak{C}}=ord\ \mathfrak{A}^{\mathfrak{B}}\times\mathfrak{A}^{\mathfrak{C}}=nm^{b+c},$$

and

$$ord\ (\mathfrak{A}^{\mathfrak{B}})^{\mathfrak{C}}=ord\ \mathfrak{A}^{[\mathfrak{C}\mathfrak{B}]}=tn^cm^{bc}.$$

The permutational equivalences are readily verified.

The rules (1)–(6) lend plausibility to the notations used here for these group operations.

3. A special case of the operation of exponentiation of two permutation groups has already appeared in the literature in connection with the number of types of Boolean functions. For Pólya [3] notes that these types are representable as the equivalence classes of collections of vertices of the n-cube with regard to its automorphism group $\mathfrak{Q}_n$. In fact Pólya ([3] footnote 7) observes that $\mathfrak{Q}_n \cong \mathfrak{S}_n[\mathfrak{S}_2]$. But deg $\mathfrak{Q}_n = 2^n$ while deg $\mathfrak{S}_n[\mathfrak{S}_2] = 2n$. Slepian [4] uses the fact that $\mathfrak{Q}_n = \mathfrak{S}_2^{\mathfrak{S}_n}$ to obtain an explicit solution for this interesting enumeration problem. The only other application of exponentiation known to us occurs in [1] where the group $\mathfrak{S}_n^{\mathfrak{S}_2}$ occurs in connection with the enumeration of a certain class of bicolored graphs.

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VOLUME IN VECTOR SPACES

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This note is concerned with the determinantal formula for the volume of a parallelepiped in a finite-dimensional real vector space. We shall show that the formula can be deduced from some very simple axioms concerning "volume." In fact we consider a nonnegative set-function μ, defined on a suitable class of sets, and we assume only that μ is additive and translation-invariant: these properties determine the value $\mu(P)$, apart from a multiplicative constant, for any parallelepiped P. The proof is quite straightforward, and involves only the affine structure of the space (the notion of length is not required). It can be used to ease a traditional difficulty in expounding the theory of Lebesgue measure for a general Euclidean space; namely the difficulty of showing that congruent sets have equal measure.

In an n-dimensional real vector space, a parallelepiped with a vertex at the origin is determined by n linearly independent vectors $x_1, \cdots, x_n$. These vectors being given, we consider the set P of all vectors $\lambda_1 x_1 + \cdots + \lambda_n x_n$, where $0 \leqq \lambda_i < 1$ for $i = 1, \cdots, n$. We denote by $\delta(P)$ the determinant of the $n \times n$ matrix (ξ_j^i), where $\xi_j^1, \cdots, \xi_j^n$ are the coordinates of x_j, for $j = 1, \cdots, n$, relative to a given basis (which we keep fixed throughout the discussion).

Suppose that $\mathfrak{R}$ is a class of sets having the following properties:

(1) P belongs to $\mathfrak{R}$, for any choice of $x_1, \cdots, x_n$, including the degenerate cases in which one of these vectors is zero;

(2) if $\mathfrak{R}$ contains sets E and H then it contains the union $E \cup H$ and the relative complement $E \setminus H$, and hence also the intersection $E \cap H$;
(3) if $\mathfrak{R}$ contains E then it contains all translates of E (that is, all sets of the type $E+x$, where x is an arbitrary vector).

Let μ be a nonnegative function defined on $\mathfrak{R}$ with the property that if E and H belong to $\mathfrak{R}$ and have empty intersection then

$$\mu(E \cup H) = \mu(E) + \mu(H).$$

Suppose also that if E belongs to $\mathfrak{R}$ and x is any vector then $\mu(E+x) = \mu(E)$. (For example, $\mathfrak{R}$ could consist of all the Borel sets, and μ could be Lebesgue measure.) We shall show that there is a number κ, independent of P, such that $\mu(P) = \kappa \left| \delta(P) \right|$.

When $i \neq j$, let P_{ij} be the set obtained by replacing x_i, in the definition of P, by $x_i + x_j$; this is equivalent to replacing λ_j by $\lambda_i + \lambda_j$. Let $A = P \setminus P_{ij}$, $B = P \cap P_{ij}$, $C = P_{ij} \setminus P$. According to our assumptions, these sets belong to $\mathfrak{R}$ and

$$\mu(P) = \mu(A \cup B) = \mu(A) + \mu(B),$$
$$\mu(P_{ij}) = \mu(B \cup C) = \mu(B) + \mu(C).$$

It is easy to see that A is the set obtained from the definition of P by imposing the restriction that $\lambda_i > \lambda_j$ (since λ_j can be expressed as $\lambda_i + \lambda_j'$, with $0 \leqq \lambda_j' < 1$, if and only if $\lambda_i < \lambda_j$). Similarly, C is the set obtained from the definition of P by replacing λ_j by $\lambda_i + \lambda_j$ and imposing the restriction that $\lambda_i + \lambda_j \geqq 1$; this is equivalent to replacing λ_j by $\lambda_j' + 1$, where $0 \leqq \lambda_j' < \lambda_i$. Thus $C = A + x_j$. Hence $\mu(C) = \mu(A)$, and therefore $\mu(P_{ij}) = \mu(P)$.

Next, for each i and each real number λ, let $P_i^{(\lambda)}$ be the set obtained by replacing x_i, in the definition of P, by λx_i. Evidently, $P_i^{(\lambda)}$ belongs to $\mathfrak{R}$ if $\lambda \geqq 0$. If p and q are positive integers then

$$P_i^{(p/q)} = P_i^{(1/q)} \cup \left(P_i^{(1/q)} + \frac{1}{q} x_i \right) \cup \cdots \cup \left(P_i^{(1/q)} + \frac{p-1}{q} x_i \right),$$

so that $\mu(P_i^{(p/q)}) = p\mu(P_i^{(1/q)})$. In particular,

$$\mu(P) = \mu(P_i^{(1)}) = q\mu(P_i^{(1/q)}).$$

Hence $\mu(P_i^{(p/q)}) = (p/q)\mu(P)$. From this it follows in an obvious way (since μ is monotonic) that $\mu(P_i^{(\lambda)}) = \lambda\mu(P)$ for any $\lambda \geqq 0$. We also have

$$(P_i^{(-1)} + x_i) \cup P_i^{(0)} = P \cup (P_i^{(0)} + x_i),$$

so that $P_i^{(-1)}$ belongs to $\mathfrak{R}$, and $\mu(P_i^{(-1)}) = \mu(P)$. Hence $P_i^{(\lambda)}$ belongs to $\mathfrak{R}$ for every value of λ, and $\mu(P_i^{(\lambda)}) = \left| \lambda \right| \mu(P)$.

It is now clear that $\mu(P)$, regarded as a function of the matrix (ξ_j^i), has the following properties: (i) its value is unaffected if any one column is added to

another; (ii) if any column is multiplied by λ then the value of the function is multiplied by $|\lambda|$. Hence if

$$f(P) = \begin{cases} \mu(P) & \text{when } \delta(P) \geqq 0, \\ -\mu(P) & \text{when } \delta(P) < 0, \end{cases}$$

then $f(P)$ has the property (i) and also the property that its value is multiplied by λ if any column of the matrix (ξ_j^i) is multiplied by λ. Now it is well known (and easy to prove*) that the only functions with these two properties are the constant multiples of $\delta(P)$. Thus, since $\mu(P) = |f(P)|$,

$$\mu(P) = \kappa |\delta(P)|,$$

where κ is the value of $\mu(P)$ when (ξ_j^i) is the unit matrix (so that $\kappa = 1$ in the case of Lebesgue measure). Since the determinant of a product of square matrices is the product of their determinants, a change of basis can only affect the value of κ.

Lebesgue measure in a Euclidean space is usually defined in terms of coverings by parallelepipeds whose edges are parallel to the vectors of a fixed orthogonal basis. It is then obvious that the measure is invariant under translations. In the light of the result we have just proved, it is almost equally obvious that the measure is invariant under general rigid displacements; or, equivalently, that it is independent of the particular orthogonal basis which is chosen for the definition. In fact it is easy to see that, more generally, if a set is subjected to any linear transformation then its Lebesgue outer measure is multiplied by the modulus of the determinant of the transformation.

*See, for example, L. Mirsky, An Introduction to Linear Algebra, Oxford, 1955, pp. 190–191.

CONVEX SETS WITH DENSE EXTREME POINTS

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Let L denote a linear space and let C be a convex subset of L. A point x_0 is called an extreme point of C if

(1) $x_0 \in C$ and

(2) no segment $\{x | x = x_0 + ty,\ -1 \leqq t \leqq 1\}$ with $0 \neq y \in L$ is contained in C.

If L has a suitable topology (*e.g.* if L is a Banach space) a celebrated theorem by Krein-Milman [1] states that if C is a compact convex set, then C is the closed convex hull of the set of its extreme points. (The closed convex hull of a set S is the smallest closed convex set containing S.)

The following problem was proposed to the author by W. G. Bade: Does there exist a convex set (other than a single point) which is the *closure* of the set of its extreme points. It appears to be widely known that the answer is affirmative, but apparently no example has been given in the literature, and therefore it seems to be worthwhile to publish the following simple construction.

THEOREM. *Let l^2 be the real Hilbert space of sequences $x=(x_1, \cdots, x_i, \cdots)$ with $\sum_{i=1}^{\infty} x_i^2 < \infty$. Let C denote the subset of l^2 determined by $\sum_{i=1}^{\infty}(2^i x_i)^2 \leqq 1$. Then C is a convex set, and C is the closure of its set of extreme points.*

Proof. Let E^n denote the subspace of l^2 determined by $x_{n+1}=x_{n+2}=\cdots=0$ and let P_n denote the projection on E^n, *i.e.*, if $x=(x_1, x_2, \cdots)$, then $P_n x=(x_1, \cdots, x_n, 0, 0, \cdots)$. Then we clearly have $C \cap E^n = P_n(C) =$ the ellipsoid determined by the inequality

$$\frac{x_1^2}{(2^{-1})^2}+\frac{x_2^2}{(2^{-2})^2}+\cdots+\frac{x_n^2}{(2^{-n})^2} \leqq 1.$$

Define $C_n = C \cap E^n$; it is then clear that when we consider C_n as a subset of E^n its extreme points are the points on the boundary B_n of C_n. Furthermore all points of B_n are also extreme points of C, for if $x_0 \in B_n$ and $0 \neq y \in l^2$ there exists an $m \geqq n$ such that $P_m y \neq 0$. Now $x_0 \in B_n \subset B_m$; hence x_0 is an extreme point of C_m and $x_0 + t \cdot P_m y = P_m(x_0 + ty)$ does not belong to $C_m = P_m(C)$ for all t with $-1 \leqq t \leqq 1$. Consequently $x_0 + ty$ does not belong to C for all t with $-1 \leqq t \leqq 1$, and, since y is arbitrary, x_0 is an extreme point of C.

Define $B = \bigcup_{n=1}^{\infty} B_n$; then B is contained in the set of extreme points of C, and since C is obviously closed the theorem will be proved when we show that B is dense in C. Let $x \in C$; then $P_n x \in C_n$ and $\|x - P_n x\| \leqq 2^{-n-1}+2^{-n-2}+\cdots = 2^{-n}$. Since the smallest semi-axis of the ellipsoid C_n has length 2^{-n}, every point of C_n has distance at most 2^{-n} from B_n. Therefore every point of C has distance at most $2^{-n}+2^{-n}=2^{-n+1}$ from B_n, and the theorem follows.

It should be noted that the above construction can be carried out in any Banach space with a basis [2]. Also note that although the set C constructed above is compact this is by no means necessary—the paraboloid $x_1 \geqq \sum_{i=2}^{\infty}(2^i x_i)^2$ is another example which is even unbounded.

A somewhat more complicated example of a convex set which is the closure of the set of its vertices [3] can also be constructed. To do this, we construct an ascending sequence of convex polyhedra $H_n \subset E^n$ with properties similar to the properties of the C_n above and such that $\bigcup_{n=1}^{\infty} V_n$ is dense in the closure of $\bigcup_{n=1}^{\infty} H_n$, where V_n denotes the set of vertices of H_n.

Added in proof. V. L. Klee, Jr., has proved that in every infinite-dimensional Banach space, "most" compact convex sets are the closure of their set of extreme points. It is planned to publish this result in a paper entitled *Some new results on smoothness and rotundity in normed linear spaces.*

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CLASSROOM NOTES

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THE LAW OF THE MEAN

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In discussing the Law of the Mean, we consider the function

$$\phi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

This is a formidable expression whose origin is puzzling until it is pointed out as the difference PQ of the ordinate of a point on the graph of $f(x)$ and the ordinate to the secant line for the same x.

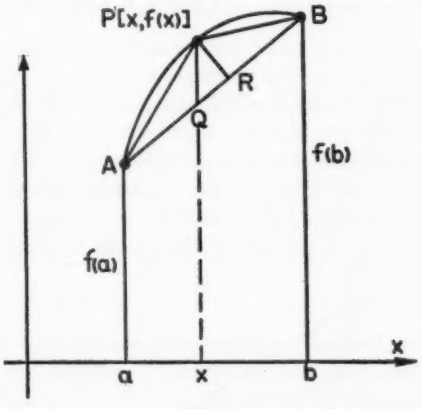

FIG. 1

A different approach results from selecting PR, the perpendicular from P to the secant (Fig. 1). Now $\sin \measuredangle PQR = PR/PQ$ and, since $\measuredangle PQR$ is a constant angle for all values x in the interval, then PR is proportional to PQ throughout and thus has all the desired properties of $\phi(x)$. But this variable quantity PR is the altitude of the triangle APB whose base is the chord AB. Thus $\phi(x)$ is also proportional to the area K of triangle APB. Accordingly, instead of $\phi(x)$, we might as well consider:

$$2K = \begin{vmatrix} x & f(x) & 1 \\ a & f(a) & 1 \\ b & f(b) & 1 \end{vmatrix}.$$

The derivative of this variable-area function must vanish by Rolle's theorem at some $x=X$ between a and b. That is,

$$\begin{vmatrix} 1 & f'(X) & 0 \\ a & f(a) & 1 \\ b & f(b) & 1 \end{vmatrix} = 0,$$

or

$$f(a) - f(b) - f'(X)(a - b) = 0.$$

Thus

$$(b - a)f'(X) = f(b) - f(a).$$

ON EQUILIBRIUM OF A TRIANGULAR PRISM FLOATING FREELY WITH ITS EDGES HORIZONTAL

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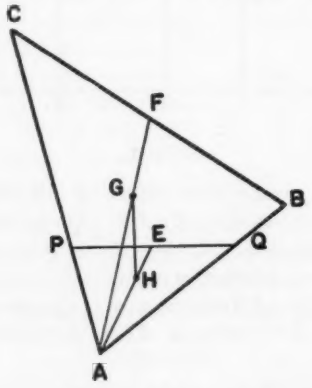
This classical problem has been discussed by Besant and Ramsey [1] and their investigation is based on the methods of ordinary algebra and geometry. In the present note the problem has been treated by the method of vector algebra. It will be noted that the method given here can be adopted with success in solving many a classical problem on equilibrium of floating bodies which fulfill the well-known conditions of flotation, *viz.*, (i) the centre of gravity of the body and that of the liquid displaced by it should be in the same vertical line and (ii) the weight of the body should equal the weight of the liquid displaced.



We shall adopt the nomenclature and the symbols as used by Besant and Ramsey.

Let ABC represent the section of the triangular prism by a vertical plane

through its centre of gravity G, PQ the line of flotation, *i.e.*, the line of intersection of the vertical plane and the horizontal plane surface of the liquid. Also, let H be the centre of gravity of the displaced liquid. Then, in view of condition (i), GH is perpendicular to PQ. Therefore the scalar product of the vectors $\overline{PQ}$ and $\overline{GH}$ is zero, *i.e.*,

$$\overline{PQ}\cdot\overline{GH} = 0. \tag{1}$$

Again, since the volumes of the prism and the liquid displaced are proportional to the areas ABC and AQP respectively, from condition (ii) we have

$$(\overline{AQ} \times \overline{AP})\rho = (\overline{AB} \times \overline{AC})\sigma, \tag{2}$$

where ρ and σ are the densities of the liquid and the prism respectively.

The vector equations (1) and (2) determine completely the different positions of equilibrium of the floating prism.

Since $\overline{PQ}=\overline{AQ}-\overline{AP}$ and $\overline{GH}=\overline{AH}-\overline{AG}$, we have from (1)

$$(\overline{AQ} - \overline{AP})\cdot(\overline{AH} - \overline{AG}) = 0$$

or,

$$(\overline{AQ} - \overline{AP})\cdot\{\tfrac{1}{2}(\overline{AQ} + \overline{AP}) - \tfrac{1}{3}(\overline{AB} + \overline{AC})\} = 0$$

or,

$$AQ^2 - AQ\cdot AB - AQ\cdot AC\cos\theta - AP^2 + AP\cdot AB\cos\theta + AP\cdot AC = 0,$$

where $\angle BAC=\theta$.

On taking $AB=2a$, $AC=2b$, $AQ=2x$, $AP=2y$, the above relation becomes

$$x^2 - y^2 - (a + b\cos\theta)x + (b + a\cos\theta)y = 0, \tag{3}$$

which is the equation of a rectangular hyperbola referred to conjugate diameters parallel to AB and AC as axes, (x, y) being the coordinates of E, the middle point of PQ. Thus the point E lies on the curve (3) when the prism floats freely in the liquid.

Also, equation (2) reduces to

$$xy = ab\sigma/\rho = c^2 \tag{4}$$

(say), which is also the equation of a rectangular hyperbola satisfied by the coordinates of the point E.

Equations (3) and (4) are the cartesian equations deduced from the vector equations (1) and (2) and are the same as obtained by Besant and Ramsey. Taken together they determine all the positions of equilibrium of the prism.

Reference

1. W. H. Besant and A. S. Ramsey, A Treatise on Hydromechanics, Pt. 1 (Hydrostatics), London, 1904, p. 45.

SOME PROPERTIES OF ARITHMETIC PROGRESSIONS

K. SUBBA RAO, Maharajah's College, Vizianagram, India

It has been stated [1] that "the product of four consecutive terms of an arithmetic progression of integers plus the fourth power of the common difference is a perfect square but in no case a perfect fourth power." It has been shown [2] by me and several others that the first part of the problem follows immediately and that the second part is false. In fact, if $(a-3d)$ be the first term and $2d$ the common difference of the A.P. of integers, we have

$$(a-3d)(a-d)(a+d)(a+3d)+16d^4=(a^2-5d^2)^2. \tag{1}$$

Now (a^2-5d^2) can be made equal to a perfect nth power, by the method of Euler and Lagrange [3]. Hence (1) can be made equal to a perfect $2n$th power.

Some analogous results are known [4] but results of a general nature on this topic seem to be new. I consider the solution of some Diophantine equations involving integers in A.P., whose first term is not an integral multiple of the common difference.

THEOREM 1. *The Diophantine equation*

$$x_1^2+\cdots+x_{2m+1}^2=(2m+1)z^n,$$

where $x_1, \cdots, x_{2m+1}$ *are integers in A.P., and* z, n *are integers, has infinitely many solutions.*

Proof. Setting $x_1=(a-3md)$, $x_2=\{a-3(m-1)d\}, \cdots$ we have

$$x_1^2+\cdots+x_{2m+1}^2=(2m+1)(a^2+kd^2), \quad \text{where} \quad k=3m(m+1).$$

Now, by the method of Euler and Lagrange cited above, (a^2+kd^2) can be made a perfect nth power and the result follows.

COROLLARY. *If* $(2m+1)$ *is a perfect square* $=q^2$ *say, then*

$$x_1^2+\cdots+x_{2m+1}^2=q^2(a^2+kd^2)=x^2+ky^2, \quad \textit{where} \quad x=qa,\ y=qd;$$

and x^2+ky^2 *can be made a perfect nth power.*

THEOREM 2. *The Diophantine equation*

$$x_1^2+x_2^2+\cdots+x_{2m}^2=2mz^n,$$

where $x_1, \cdots, x_{2m}$ *are integers in A.P. and* z, n *are integers, has infinitely many solutions.*

Proof. Setting $x_1=\{a-3(2m-1)d\}$, $x_2=\{a-3(2m-3)d\}, \cdots$, and proceeding as in 1 above, we prove the result.

COROLLARY. *If $2m$ is a perfect square, then $\sum_1^{2m} x_i^2$ can be made a perfect nth power.*

THEOREM 3. *The Diophantine equation*

$$x_1^2 + \cdots + x_m^2 = z^{2n+1},$$

where $x_1, \cdots, x_m$ are integers in A.P. and z, n are integers, has infinitely many solutions.

Proof. I consider two cases.

Case I. Let m be even and $=2p$.

Let $x_1 = \{a-(2p-1)3d\}$, $x_2 = \{a-(2p-3)3d\}$ $\cdots$. Then $x_1^2 + \cdots + x_m^2 = 2p(a^2+kd^2)$, where

$$k = 3(4p^2 - 1).$$

Set $a = (2p)^n x(x^2+ky^2)^n$, $d = (2p)^n y(x^2+ky^2)^n$, where x and y are integers and $(x, y) = 1$. Then $\sum_1^m x_i^2 = z^{2n+1}$, where $z = 2p(x^2+ky^2)$, and the theorem is proved.

Case II. Let m be odd and $=(2p+1)$. By proceeding as in Case I, the result follows.

THEOREM 4. *The Diophantine equation*

$$x_1^3 + \cdots + x_m^3 = z^n,$$

where $x_1, \cdots, x_m$ are integers in A.P., z an integer and n is an integer of the form $(3s+1)$ or $(3s+2)$, has infinitely many solutions.

Proof. If m be even and $=2p$, by setting

$$x_1 = \{a - (2p - 1)d\}, \qquad x_2 = \{a - (2p - 3)d\}, \cdots,$$

we have $x_1^3 + \cdots + x_m^3 = 2p \cdot a(a^2+kd^2)$, where $k = 4p^2 - 1$. If we put $a = x(x^2+ky^2)^s$, $d = y(x^2+ky^2)^s$, where x, y, s are integers and $(x, y) = 1$, we have

$$\sum_1^m x_i^3 = 2px(x^2 + ky^2)^{3s+1}.$$

On replacing x by $(2p)^{3s}x^{3s+1}$, we can make

$$\sum_1^m x_i^3 = \text{the } (3s+1)\text{th power of an integer.}$$

Again, setting $a = x(x^2+ky^2)^{2s+1}$, $d = y(x^2+ky^2)^{2s+1}$ and proceeding as above, we can make

$$\sum x_i^3 \text{ equal to the } (3s+2)\text{th power of an integer.}$$

The method is similar, when m is odd.

Note. Some special cases of this result, when $n=2, 3, 4$, are to be found in Chapters XXI and XXII of [4].

References

1. E876, this MONTHLY, vol. 56, 1949, p. 473.
2. E876, this MONTHLY, vol. 57, 1950, pp. 186–187.
3. L. E. Dickson, Introduction to the Theory of Numbers, University of Chicago Press, 1929, 96–97.
4. ———, History of the Theory of Numbers, Washington, 1919, pp. 320–323.

MATHEMATICAL EDUCATIONAL NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

WHAT TO DO ABOUT A NEW KIND OF FRESHMAN*

J. H. NEELLEY, Carnegie Institute of Technology

At Carnegie Tech last fall we had our first freshmen who had had a calculus course in high school. They were generally bright young people, but tests showed that they merely had some formulas. They knew not the why of calculus. However, some were permitted to enter second semester courses in analytics and calculus. This we did not like to do. It was bad, very bad, to put these bright ambitious students in classes with the failures of the previous year. They had to get a completely wrong idea of college work from these flunkers.

Others were not permitted to omit the first semester. Such decisions were made after the students were given individual examinations. These too were unhappy as they had been told in high school that they would have advanced credit. The fact is that no such student gets any college credit at Carnegie Tech. Their high school courses are of the simplest possible types, due to teachers of little experience in higher mathematics and to the choices of the most innocuous texts available. Hence no credit can be given. Those who start in the second semester courses merely have to take something to replace the hours toward a degree which were omitted.

At first those who were not given advanced standing were sure they had not gotten a square deal. Then they came to me later in the semester with statements

* Presented to Allegheny Mountain Section of M.A.A., May, 1958.

like, "I now know why and not merely how," or "I would have missed a lot if I had been allowed to start with the second semester." It seems to me that such statements are very significant. They definitely point to the possible futility of high school-taught analytics and calculus.

Another problem which we of the colleges face is the question of where to put such students when they offer the College Board examinations as reason for advanced standing. This is because such examinations cut across college courses rather than cover one or two courses. Any place will mean loss of what can be very important topics and ideas.

Since many colleges and universities have shifted total responsibility to high schools for all algebra and trigonometry, we are assuming that the students acquire sufficient knowledge of these subjects to permit them to handle college courses in analytics and calculus—a rather violent assumption. I say this because from 12% to 20% of our freshmen fail their first course in college mathematics, and this might be much higher if we, like many, did not use fellows, graduate students and low-ranking members of the staff as freshmen instructors. Some such instructors pass as many as they possibly can to make their records look good. However, this practice merely makes the percentage of failures in the sophomore courses higher.

The college administrations seem to think that approximately 12% is as high as should be failed. This, of course, is based on the care with which students are admitted. Really there is too little care in choice of entrants. Most institutions have more apply than they should take, but they accept them until the quotas are filled. This is done because delay in accepting a student means that he had been accepted at one of the other places of application. Hence we do not pick—the quota must be filled. So, we take them and fail them and everybody is disturbed about it, but cannot change the situation.

A study of our courses and failures over the past three years has pointed particularly to poor algebra and trigonometry as the fundamental reasons. Hence I have worked out a list of topics in mathematics for high school students who plan to go to college.* A study of this list will certainly point out that the high schools have enough to do without attempting to teach college subjects.

My sincere conclusion on this problem is that the colleges and universities should test any student presenting analytics or calculus and be very critical of his right to advanced standing. I urge that no credit be given and that advanced standing be granted very rarely and that in such cases a close check be kept on such students. I urge you to discourage the public schools in your areas in teaching analytics and calculus. Point out to them the high casualty list of their students as of now; propose more algebra and analytic trigonometry, instead of that deadening drudgery of solutions of triangles by all means and methods, ad infinitum and ad nauseam.

I propose and urge you of the secondary schools to take this advice and come

* The Mathematics Teacher, December, 1958.

up with a better prepared group than you have been able to do up to now. Perhaps you do not know that many colleges and universities must give pre-freshman courses in mathematics to many entering students. Some even give pre-pre-freshman courses.

Finally, I quote some figures given out recently by one of our state universities. Only one third of the applicants were able to pass tests given at the university to decide whether to place them in college mathematics or not. *One fifth* had to take pre-college courses. *One fourth* had to take pre-pre-college courses. This is appalling, and here is what a paper put out by the university has to say about it. "Without affixing blame anywhere, let's regard some of today's children as non-educable on the college level and create junior colleges or institutions where they can either learn less expensive and lower-grade skills and cultures, or work their way up to the universities. . . . With a doubling of the university's enrollment by 1970, the 'spoon feeding of illiterates' will be a luxury we can no longer afford."

I sincerely hope that all levels of education can come to agree with me on this situation and bring about a more happy situation in the near future.

CHEMISTRY ON FILM

JOHN F. BAXTER, University of Florida

A project aimed at putting a complete course in high school chemistry on film has just been completed at the University of Florida. The author of this report, who is the "film teacher," is Professor of Chemistry and Head of the Division of General Chemistry at the University of Florida. Following the success of the course in Physics, taught by Dr. Harvey White of the University of California, as a "live" TV course in Pittsburgh during the school year 1956–57, and filmed in color by Encyclopaedia Britannica Films, Inc., plans were begun to film a course in chemistry. The Fund for the Advancement of Education, which supported the physics project, approached the American Chemical Society (ACS) about a course in chemistry. I was selected to plan and teach the course, with the cooperation and advice of a seven-member committee representing the ACS, the ACS Division of Chemical Education, and the National Science Teachers Association. Like the physics course, the chemistry films are being produced and distributed by the Encyclopaedia Britannica Films.

The original plans called for about 160 lessons of 30 minutes each, to cover a year's work. After the project was begun it was decided to make possible greater flexibility in the use of the course by building a core of 130–135 sequential lessons. The remainder of the lessons can be used to supplement and enrich the presentation, need not be used in sequence, and, if necessary because of time limitations, one or more may be omitted without destroying the continuity of the course. Experience with the physics films had shown that the inroads of special holidays, assemblies, football rallies, and the other distractions of what seems to have become the normal high school program, often left fewer than

160 days for classroom teaching in a year's course. Furthermore, it was the conviction of those planning the chemistry course, that it was desirable to encourage use of laboratory time by the student in his own school where laboratory teachers and facilities were available.

Correlated to each lesson, student manuals and teacher's manuals have been prepared to supplement the use of the films. For the student, preliminary information is given as to "what to look for" in each film. Questions directly related to the filmed lesson are provided. In addition, questions and problems related to the lesson but not actually covered in the films, provide additional stimulation for the interested student. The teacher's manual includes a detailed "time sequence" of topics and demonstrations for the use of the teacher in preparing for each lesson. It also provides answers to questions in the student manual, together with suggestions for experiments, projects, and related readings for the student. No current high school text is followed in the film and it is felt that the filmed course, together with the manuals, provides the essentials of a complete course in chemistry. However, the use of a standard text for supplementary purposes is encouraged and each lesson is keyed by references to 16 currently-used high school texts.

The course is reasonably conventional in its approach to chemistry, but there is much more emphasis on fundamental principles of chemistry, with descriptive features of chemistry being discussed in terms of these principles. An earnest attempt has been made to bring chemistry up to date. One of the objections to many texts being currently used is that they are rather sadly out of date in their approach to much of chemistry, although there are noteworthy exceptions.

The project had as one of its key objectives at the beginning, the making available of a course in chemistry in schools where it was not being taught, or where there may not have been a teacher with adequate training in chemistry. However, it is believed that one of its most significant uses may be in taking over much of the impossible drudgery heaped on the capable teacher who may be teaching five classes in two or more sciences and who simply does not have time to plan and prepare class demonstrations or to cope with the physical problems of keeping a laboratory operating. The filmed course is rich in demonstrations and "visuals" of many kinds. Its use can make time available to the experienced teacher to keep a really top-flight laboratory in operation and to plan superior classroom teaching. It is now felt by many who have watched the course develop that perhaps the most significant use of all might be in teacher-training institutions.

The films are already in use in high schools, both in classroom projection, and on open channel TV.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1376. *Proposed by V. F. Ivanoff, San Carlos, California*

Show that if A is the area of a quadrilateral having sides a, b, c, d and diagonals e, f then

$$16A^2 = 4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2.$$

E 1377. *Proposed by D. J. Newman, AVCO Research and Development, Lawrence, Mass.*

Solve the system of equations

$$u + v = a, \qquad ux + vy = b, \qquad ux^2 + vy^2 = c, \qquad ux^3 + vy^3 = d$$

for u, v, x, y.

E 1378. *Proposed by Paul Pargas, Washington, D. C.*

Draw a circle by tracing around a given circular disk D and mark an arbitrary point A on the circle. By drawing circles only with the use of the disk D, where one is permitted to place the rim of D on any two distinct given points, find the point B on the original circle which is diametrically opposite the point A.

E 1379. *Proposed by A. A. Mullin, University of Illinois*

Let S be the set of all real numbers and let $a * b$ denote $\max(a, b)$, where $a \in S$, $b \in S$. Show that the system $\{S; *\}$ is an abelian semigroup.

E 1380. *Proposed by D. S. Passman, Polytechnic Institute of Brooklyn*

Let A and B be $n \times n$ nonsingular matrices with $A + B = kE$, where k is a scalar and E is a matrix all of whose elements are 1's. If $S(C)$ denotes the sum of the elements of C, show that

$$\{1 - kS(A^{-1})\}\{1 - kS(B^{-1})\} = 1.$$

SOLUTIONS

A Mechanical Quadrature Formula

E 1345 [1958, 775]. *Proposed by I. J. Schoenberg, University of Pennsylvania*

There are given $2n$ points on the x-axis: $x_1 < \cdots < x_{2n}$, $(n \geq 1)$. For convenience we also write $x_0 = x_1$, $x_{2n} = x_{2n+1}$ and determine, for each $k = 1, \cdots, n$,

a point ξ_k between x_{2k-1} and x_{2k} satisfying the linear equation

$$(\xi_k - x_{2k-2})(\xi_k - x_{2k-1}) = (\xi_k - x_{2k})(\xi_k - x_{2k+1}),$$

thus obtaining n points $\xi_1 < \cdots < \xi_n$. Finally, let

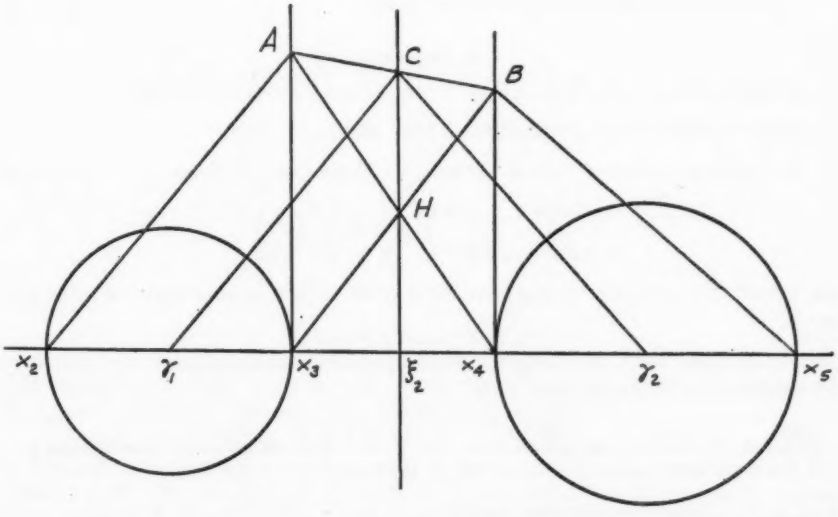
$$\gamma_k = (x_{2k} + x_{2k+1})/2, \qquad k = 0, 1, \cdots, n.$$

Show that the relation

$$\sum_{k=1}^{n} (\gamma_k - \gamma_{k-1}) f(\xi_k) = \int_{x_1}^{x_{2n}} f(x)dx \tag{1}$$

holds for any continuous function $f(x)$ which is linear in each of the intervals (x_k, x_{k+1}), $k = 1, \cdots 2n-1$.

Solution by the proposer. It suffices to prove that (1) holds for each of the $2n$ "roof-functions" obtained by putting $f(x_k) = 0$ for all k, with the exception of just *one* such value. For indeed, an arbitrary broken line $f(x)$ may be represented as an appropriate linear combination of such roof-functions. The case of the two roof-functions corresponding to $f(x_3) > 0$ and $f(x_4) > 0$, respectively, is typical. The relation (1) for each of these functions follows from a theorem in elementary geometry illustrated by the accompanying figure.



The line $\xi_2 H$ is the radical axis of the circles having centers γ_1 and γ_2. H being a point of this axis, produce the lines x_3H, x_4H, and let them intersect at A and B the tangents to the circles at the points x_3 and x_4 respectively. Let AB intersect the axis at C.

THEOREM. *The three triangles* x_2Ax_4, $\gamma_1C\gamma_2$, *and* x_3Bx_5 *all have the same area.*

We omit the simple proof, which can be carried out in several ways. Observing that $\xi_2H=HC$, the equality of the areas of the first two triangles may be written as

$$(\gamma_2-\gamma_1)(\xi_2H)=\text{area of triangle } x_2Ax_4,$$

and this is precisely the formula (1) for the roof-function whose graph is the polygonal line $x_1x_2Ax_4x_{2n}$. The case of the first two roof-functions ($f(x_1)>0$ and $f(x_2)>0$) is also covered by the geometric theorem if the left circle is allowed to shrink to a point.

Formula (1) shares with Gauss's quadrature formula the property of giving the integral of each function of a $2n$-parameter family from its ordinates at only n fixed abscissae. Noteworthy is the case of equidistant points $x_i=i-1$, $i=1, \cdots, 2n$, when (1) becomes

$$(3/2)f(2/3)+2\sum_{k=1}^{n-2}f(2k+1/2)+(3/2)f(2n-5/3)=\int_0^{2n-1}f(x)dx.$$

This formula holds for any broken line function with vertices at integral values of x.

Also solved by D. C. B. Marsh and David Zeitlin.

A Congruence

E 1346 [1959, 61]. *Proposed by J. M. Gandhi, Belgaum, India*

Prove that if p is a prime then $\binom{2p}{p}\equiv 2$, mod p.

I. *Solution by David Zeitlin, Remington Rand Univac.* Since

$$\binom{2p}{p}=\sum_{k=0}^{p}\binom{p}{k}^2=2+\sum_{k=1}^{p-1}\binom{p}{k}^2,$$

the result follows upon noting that for p prime, $\binom{p}{k}/p$ is an integer, $k=1, \cdots, p-1$.

II. *Solution by J. H. Hodges, Cornell Aeronautical Laboratory, Inc.* Let k be an arbitrary positive integer. Then

$$\binom{kp}{p}\equiv\frac{(kp)!}{p!(kp-p)!}\equiv\frac{k(kp-1)\cdots(kp-p+1)}{(p-1)!}\equiv\frac{k(p-1)!}{(p-1)!}\equiv k, \text{ mod } p.$$

Taking $k=2$ gives the desired result.

III. *Solution by T. M. Little, University of California.* In the expansion of $(1+1)^{2p}=2^{2p}=4^p$, every term except the first, middle, and last is divisible by p. Therefore $4^p\equiv 2+\binom{2p}{p}$, mod p. By Fermat's theorem, $4^p\equiv 4$, mod p. So $\binom{2p}{p}\equiv 2$, mod p.

IV. *Solution by Leo Moser, University of Alberta.* Consider the $\binom{2p}{p}$ arrangements of p white and p black beads in a circular necklace, with the necklace fixed in position. There are two arrangements in which the white and black beads are interlaced and since p is a prime the remaining arrangements come in sets of $2p$ by rotation. Hence $\binom{2p}{p} \equiv 2 \pmod{2p}$, a slightly stronger result than the one required.

A similar argument reveals that, more generally, for p and q primes,

$$(qp)!/(p!)^q \equiv q! \pmod{pq}.$$

V. *Solution by L. D. Goldstone, New York State Public Works Lab.* In 4322 [1950, 347] it is shown that if p is prime, then $\binom{n}{p} \equiv [n/p] \pmod{p}$, where $[n/p]$ indicates the largest integer contained in n/p. Since $[2p/p] = 2$, it then follows that $\binom{2p}{p} \equiv 2 \pmod{p}$.

VI. *Solution by N. G. Gunderson, University of Rochester.* Actually, if $p > 3$, $a > 0$, then $\binom{ap^n}{p} \equiv ap^{n-1} \pmod{p^{2n+1}}$. Thus for $p > 3$, $\binom{2p}{p} \equiv 2 \pmod{p^3}$. For $p = 2$, $\binom{4}{2} \equiv 2 \pmod{2^2}$.

Letting S_j represent the sum of the products of $1, \cdots, p-1$ taken j at a time, and letting T be defined by

$$T(p-1)! \equiv 1 \pmod{p^{2n+1}},$$

we have

$$\begin{aligned}\binom{ap^n}{p} &= (ap^n)(ap^n - 1) \cdots (ap^n - p + 1)/p! \\ &\equiv ap^{n-1}\{1 - TS_{p-2}ap^n + TS_{p-3}a^2p^{2n}\} \pmod{p^{2n+1}}.\end{aligned}$$

However,

$$\begin{aligned}(p-1)! &= \{p - (p-1)\}\{p - (p-2)\} \cdots \{p - 1\} \\ &\equiv (p-1)! - pS_{p-2} + p^2S_{p-3} \pmod{p^3},\end{aligned}$$

and so

$$pS_{p-2} \equiv p^2S_{p-3} \pmod{p^3}.$$

But $S_{p-3} \equiv 0 \pmod{p}$ from $(x-1) \cdots (x-p+1) \equiv x^{p-1} - 1 \equiv 0 \pmod{p}$, and so $S_{p-2} \equiv 0 \pmod{p^2}$, and the first stated result follows.

Also solved by A. N. Aheart, R. G. Albert, J. L. de M. Arenal, Anders Bager, H. F. Bechtell, G. E. Bloch, J. L. Botsford, D. A. Breault, R. F. Brown and Anna Endelman and Joel Levy (jointly), J. M. Calloway, Bomshik Chang, G. B. Charlesworth, P. L. Chessin, A. E. Danese, F. J. Duarte, E. S. Eby, N. J. Fine, Brother Louis Francis, D. A. Freedman, Rheba Galloway, Michael Goldberg, Sidney Glusman, Mildred Gross, Joseph Hammer, L. H. Hauer, S. A. Hoffman, J. Hooley, J. M. Howell, E. M. Horadam, J. T. Humphrey, M. S. Itzkowitz, M. I. Knopp and E. M. Scheuer (jointly), Sidney Kravitz, K. L. Loewen, D. C. B. Marsh, Helen Marston, C. E. Miller, Franklin Mohr, D. L. Muench, J. B. Muskat, Stewart Nagler, K. K. Norton, C. S. Ogilvy, C. Oster, F. D. Parker, Erna Pearson, D. J. Persico, Stanton Philipp, J. L. Pietenpol, C. F. Pinzka,

E. J. F. Primrose, Susan Pyeatt, T. L. Reynolds, L. A. Ringenberg, M. J. Saadaldin, Benjamin Sapolsky, Y. S. Sathe, L. J. Schneider, R. E. Shafer, Robert Spira, R. H. Sprague, E. P. Starke, Gerald Stoller, W. R. Talbot, R. G. Thompson and Mrs. R. G. Thompson (jointly), W. F. Trench, W. A. Veech, Harry Weingarten, Charles Wexler, Clement Winston, Dale Woods, and the proposer.

The problem is a special case of Prob. 4, p. 157, *Elementary Number Theory*, Uspensky and Heaslet, 1st ed. (1939). It is also a consequence of E 435 [1941, 269]. For numerous allied references see Dickson, *History of the Theory of Numbers*, vol. 1, pp. 270–278, in particular the reference under A. Cunningham on p. 274.

Fibonacci Numbers

E 1347 [1959, 61]. *Proposed by V. F. Ivanoff, San Carlos, California*

Prove that $\sum_{i=0}^{n} \binom{n}{i} F_{n-i} = F_{2n}$, where $F_0, F_1, \cdots, F_n$ is any set of $n+1$ consecutive Fibonacci numbers.

I. *Solution by E. P. Starke, Rutgers University.* By the difference relation of the Fibonacci numbers we have $F_{2n} = F_{2n-1} + F_{2n-2}$, which is the case $k=1$ of

$$F_{2n} = F_{2n-k} + \binom{k}{1} F_{2n-k-1} + \binom{k}{2} F_{2n-k-2} + \cdots + F_{2n-2k}.$$

For an induction hypothesis assume this equation, replace each F_j on the right by $F_{j-1} + F_{j-2}$ and replace $\binom{k}{j} + \binom{k}{j-1}$ by $\binom{k+1}{j}$. The result, after obvious reductions, is

$$F_{2n} = F_{2n-(k+1)} + \binom{k+1}{1} F_{2n-(k+1)-1} + \cdots + F_{2n-2(k+1)},$$

and the induction is complete. The proposed formula is the case $k=n$.

II. *Solution by N. J. Fine, Institute for Advanced Study.* Let f_j be the jth Fibonacci number, and let the operator E be defined by $Ef_j = f_{j+1}$. The required result is

$$(1 + E)^n f_j = f_{j+2n}.$$

By definition, it holds for $n=1$ and all j. Assuming that it holds for $n=k$, we have

$$(1 + E)^{k+1} f_j = (1 + E)(1 + E)^k f_j = (1 + E) f_{j+2k} = f_{j+2k+2}.$$

Thus it holds for $n=k+1$ and the induction is complete.

III. *Solution by David Zeitlin, Remington Rand Univac.* Since

$$F_k = c_1[(1 + \sqrt{5})/2]^k + c_2[(1 - \sqrt{5})/2]^k, \qquad k = 0, 1, \cdots$$

we have

$$\sum_{i=0}^{n} \binom{n}{i} F_{n-i} = c_1[(3 + \sqrt{5})/2]^n + c_2[(3 - \sqrt{5})/2]^n = F_{2n}.$$

Also solved by A. N. Aheart, R. G. Albert, J. L. de M. Arenal, R. J. Beeber, R. F. Brown and Anna Endelman and Joel Levy (jointly), J. M. Calloway, Leonard Carlitz, P. L. Chessin, A. E. Danese, David DeVol, Merritt Elmore, E. A. Fay, Michael Goldberg, Cornelius Groenewoud, Leonard Hauer, J. H. Hodges, S. A. Hoffman, Vern Hoggatt, J. Hooley, A. F. Horadam, A. S. Howard, H. Jager, P. G. Kirmser, M. I. Knopp and E. M. Scheuer (jointly), A. G. Konheim, T. M. Little, D. C. B. Marsh, K. K. Norton, C. S. Ogilvy, F. R. Olson, Stanton Philipp, J. L. Pietenpol, E. J. F. Primrose, B. E. Rhoades, Benjamin Sapolsky, R. E. Shafer, Ruth G. Smith, Robert Spira, R. H. Sprague, Gerald Stoller, R. G. Thompson and Mrs. R. G. Thompson (jointly), W. F. Trench, Peter Treuenfels, W. A. Veech, Julius Vogel, Everett Walter, and the proposer.

Equilateral Triangles Inscribed in an Ellipse

E 1348 [1959, 62]. *Proposed by M. S. Klamkin and Raphael Miller, AVCO Research and Development, Lawrence, Mass.*

Find the locus of the centroids of all equilateral triangles inscribed in an ellipse.

Solution by Sister Mary Stephanie, Georgian Court College, Lakewood, New Jersey. The problem is not new; it appears on page 170 of C. Smith, *Conic Sections*, Macmillan (1937). The solution given there is essentially as follows. Let the ellipse be $x^2/a^2+y^2/b^2=1$. If the eccentric angles of the vertices of an inscribed triangle are A, B, C, the centroid of the triangle is given by

$$x = a(\cos A + \cos B + \cos C)/3,$$
$$y = b(\sin A + \sin B + \sin C)/3.$$

The circumcenter of the triangle is given by

$$x = (a^2 - b^2)\{\cos A + \cos B + \cos C + \cos (A + B + C)\}/4a,$$
$$y = (b^2 - a^2)\{\sin A + \sin B + \sin C - \sin (A + B + C)\}/4b.$$

Since in an equilateral triangle the centroid coincides with the circumcenter, in this case we have

$$4ax/(a^2 - b^2) - 3x/a = \cos (A + B + C),$$
$$4by/(b^2 - a^2) - 3y/b = -\sin (A + B + C).$$

Squaring and adding we obtain the ellipse

$$(a^2 + 3b^2)^2x^2/a^2 + (b^2 + 3a^2)^2y^2/b^2 = (a^2 - b^2)^2$$

as the required locus.

Also solved by A. E. Currier, Stanton Philipp, and P. D. Thomas.

Cos $n\theta$ as a Tridiagonal Determinant

E 1349 [1959, 62]. *Proposed by P. L. Chessin, University of Maryland*

Consider the $n \times n$ matrix $[a_{ij}]$ where $a_{11}=\cos\theta$, $a_{ii}=2\cos\theta$, $(i=2, \cdots, n)$, $a_{i,i+1}=a_{i+1,i}=1$ $(i=1, \cdots, n-1)$, and all other elements are zero. Show that $|a_{ij}| = \cos n\theta$.

Solution by D. A. Breault, Sylvania Electric Products, Inc., Somerville, Mass. Denote the $n \times n$ determinant by D_n. Clearly $D_1 = \cos\theta$ and $D_2 = \cos 2\theta$. Assume $D_{k-1} = \cos(k-1)\theta$ and $D_k = \cos k\theta$, and expand D_{k+1} by minors with respect to the last row to obtain

$$D_{k+1} = 2D_k \cos\theta - D_{k-1} = 2\cos\theta\cos k\theta - \cos(k-1)\theta$$
$$= \cos\theta\cos k\theta - \sin\theta\sin k\theta = \cos(k+1)\theta.$$

The desired result now follows by mathematical induction.

Also solved by R. G. Albert, J. L. de M. Arenal, R. J. Beeber, H. F. Bechtell, Marvin Blum, A. P. Boblett, Louis Brand, J. L. Brenner, R. F. Brown and Anna Endelman and Joel Levy (jointly), A. W. Brunson, Leonard Carlitz, G. B. Charlesworth, George Cherlin, A. E. Danese, E. Desautels, E. S. Eby, Merritt Elmore, H. E. Fettis, J. F. Foley, D. A. Freedman, D. P. Giesy, Michael Goldberg, Bernard Greenspan, D. S. Greenstein, J. D. Haggard, Franklin Haimo, J. H. Hodges, S. A. Hoffman, A. F. Horadan, A. R. Hyde, P. G. Kirmser, A. G. Konheim, M. M. Levine and P. F. Zweifel (jointly), K. L. Loewen, W. R. McEwen, D. C. B. Marsh, D. S. Mitrinović, J. B. Muskat, C. S. Ogilvy, C. Oster, N. T. Peck, J. M. Perry, D. J. Persico, Stanton Philipp, E. J. F. Primrose, Susan Pyeatt, T. L. Reynolds, B. E. Rhoades, R. A. Rosenbaum, H. D. Ruderman, Benjamin Sapolsky, E. M. Scheuer, R. E. Shafer, R. H. Sprague, E. P. Starke, T. J. Syson, R. C. Thompson, Peter Treuenfels, W. A. Veech, Dale Woods, Chih-yi Wang, David Zeitlin, and the proposer.

Rosenbaum showed that the conclusion follows if the assumption $a_{i,i+1} = a_{i+1,i} = 1$ is replaced by the weaker condition $a_{i,i+1}a_{i+1,i} = 1$. The given problem occurs as Prob. 6, page 84, in Rev. E. M. Radford's *Mathematical Problem Papers*, Cambridge University Press (1923), and as Prob. 12, page 191, in D. S. Mitrinović's *Zbornik matematičkih problema*, vol. 1, 2nd ed., Belgrade (1958).

A Homothecy

E 1350 [1959, 62]. *Proposed by N. A. Court, University of Oklahoma*

(a) The tangents to the ninepoint circle of a triangle T at the midpoints of the sides of T form a triangle homothetic to the orthic triangle of T. (b) The homothetic center of the two triangles is a point on the Euler line of T.

I. *Solution by H. E. Fettis, Wright-Patterson Air Force Base, Dayton, Ohio.* Let the vertices of the triangle T be A_1, A_2, A_3, the midpoints of the sides be O_1, O_2, O_3, the feet of the altitudes be H_1, H_2, H_3, and the intersections of the tangents to the ninepoint circle at the midpoints of the sides be P_1, P_2, P_3. It is known (see Johnson, *Modern Geometry*, art. 253e) that A_1O is perpendicular to H_2H_3, where O is the circumcenter of T. Also, A_1O is parallel to O_1F, where F is the ninepoint center of T (ibid, art. 308). Therefore P_2P_3 is parallel to H_2H_3, so that triangles $H_1H_2H_3$ and $P_1P_2P_3$ are homothetic. Further, H, being the incenter of $H_1H_2H_3$, is in correspondence with F, the incenter of $P_1P_2P_3$, so that the join of these two points, namely the Euler line of T, must pass through the homothetic center of $H_1H_2H_3$ and $P_1P_2P_3$.

II. *Solution by G. B. Charlesworth, Hofstra College.* The tangential triangle of T and the orthic triangle of T are homothetic, with center of homothecy on the Euler line of T (Altshiller-Court, *College Geometry*, art. 205, p. 102). Also, if T'

is the triangle whose vertices are the midpoints of the sides of T, then T and T' are homothetic, with the centroid G of T as center of homothecy. It follows that the tangential triangles of T and T' are homothetic with G as center of homothecy. We now conclude (by art. 56, p. 18, of Altshiller-Court, *Modern Pure Solid Geometry*) that the tangential triangle of T' and the orthic triangle of T are homothetic, with center of homothecy on the Euler line of T.

Also solved by J. W. Clawson, J. F. Darling, L. D. Goldstone, D. C. B. Marsh, W. R. Talbot, Victor Thébault, and the proposer.

Thébault stated that the ratio of homothecy is 4 cos A cos B cos C, where A, B, C are the angles of triangle T. Darling used complex coordinates.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4857. *Proposed by I. N. Baker, University of Alberta, and D. J. Newman, AVCO Research, Wilmington, Mass. (independently).*

What is the most general entire function which takes real values on the real axis only?

4858. *Proposed by J. A. Riley and Theodore Hatcher, Parke Mathematical Laboratories, Carlisle, Mass.*

Define the binary capacity, $C(n)$, of a positive integer n to be the exponent of the highest power of 2 which divides n.

If $\binom{n}{h}$ is the ordinary binomial coefficient, it is known that

$$C\binom{n}{h} = l(h) + l(n-h) - l(n),$$

where $l(n)$ is the number of 1's in the binary representation of n. What is

$$C\left(\sum_{j=0}^{h}\binom{n}{j}\right)?$$

4859. *Proposed by Richard Bellman, The RAND Corporation*

Consider the Fredholm integral equation

$$u(x) = v(x) + \int_a^1 k(x, y)u(y)dy, \qquad a < 1,$$

with the solution

$$u(x) = v(x) + \int_a^1 K(x, y, a)v(y)dy.$$

Under appropriate conditions on $k(x, y)$ show that the resolvent kernel satisfies the equation

$$\frac{\partial K}{\partial a}(x, y, a) = -K(x, a, a)K(a, y, a).$$

4860. *Proposed by Immanuel Marx, Purdue University*

Given the polynomial $F_n(x) = x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^n a_n$,

and integers p and q, $0 < q \leqq p \leqq n$. From each subset of the roots of F_n taken p at a time suppose formed the elementary symmetric function of degree q, *i.e.*, the sum of all products of distinct members of the subset taken q at a time. Knowing only the coefficients of F_n, devise a method to construct a polynomial whose roots are the $\binom{n}{p}$ symmetric functions described. Illustrate the method by deriving from the quartic $F_4(x) = x^4 - a_1x^3 + a_2x^2 - a_3x + a_4$, having (unknown) roots r_1, r_2, r_3, r_4, the quartic whose roots are $R_1 = r_2r_3 + r_3r_4 + r_4r_2$, $R_2 = r_1r_3 + r_3r_4 + r_4r_1$, $R_3 = r_1r_2 + r_2r_4 + r_4r_1$. $R_4 = r_1r_2 + r_2r_3 + r_3r_1$; in other words, the case $n = 4$, $p = 3$, $q = 2$.

4861. *Proposed by Joseph Lehner, Michigan State University*

Find a function $f(x)$, nonnegative and bounded on the closed interval $[0, \infty]$, such that

$$\limsup_{t \to \infty} \left| \int_0^t f(x)e^{2\pi\lambda x}dx \right| = +\infty$$

for all real λ with the exception of a countable set.

4862. *Proposed by D. J. Newman, AVCO Research, Wilmington, Mass.*

Given a set S of distinct numbers $a_1, \cdots, a_n$. Define $k = k(S)$ to be the number of numbers of the form $a_i + 2^j$, $j = 1, 2, \cdots, n$. What is $\min_S k(S)$?

SOLUTIONS

The Jacobi Symbol

4809 [1958, 633]. *Proposed by Sylvan H. Greene, General Electric Co., Philadelphia*

Let n be an odd, square-free integer greater than 3; let A_n be the set of all a (mod n) such that $(a/n) = +1$, and B_n that of all b (mod n) such that (b/n)

$=-1$, where (b/n) is the Jacobi symbol. Prove that in a complete system of residues (mod n), $\sum a \equiv \sum b \equiv 0 \pmod{n}$. The summations are taken over all elements of A_n and B_n respectively.

Solution by Leonard Carlitz, Duke University. Because of the hypothesis on n, there exists a number $c \not\equiv 1 \pmod{n}$ such that $(c/n) = 1$. [Proof. Let p be a prime >3 that divides n, whence $(n, n/p) = 1$. Let r be an integer such that $r \not\equiv 1 \pmod{p}$ and $(r/p) = 1$. Then determine c by means of $c \equiv r \pmod{p}$, $c \equiv 1 \pmod{n/p}$.] Then if $A = \sum a$, $B = \sum b$, it follows that

$$A \equiv \sum_a ca \equiv cA, \qquad B \equiv \sum_b cb \equiv cB,$$

whence $A \equiv 0$, $B \equiv 0 \pmod{n}$.

Also solved by N. Sugunamma and by the proposer.

Convex Curves

4810 [1958, 712]. *Proposed by G. B. Dantzig and L. R. Ford, Jr., The Rand Corporation*

Suppose that C_1 and C_2 are convex curves with continuously turning tangents, and that the tangents at the left endpoints are parallel, as are the tangents at the right endpoints. Let C be the locus of points which are the midpoints of line segments joining a point on C_1 with a point on C_2 where the tangents are parallel.

Show that: (a) C is also convex; (b) the midpoint of any segment joining a point on C_1 to a point on C_2 cannot lie below C; and (c) if C_1 and C_2 are placed on parallel planes in 3-space, the lines joining points with parallel tangents generate a convex surface.

Solution by P. C. Hammer, University of Wisconsin. This problem is readily seen to be a special application of more general and well-known results. Let B_0 and B_1 be closed bounded convex sets in E_n. Treating points as vectors, let $0 \leqq t \leqq 1$ and define $B_t = (1-t)B_0 + tB_1$. Here the $+$ refers to the direct sum. Then it is quickly established that B_t is a bounded closed convex set in E_n and the union of all B_t, $0 \leqq t \leqq 1$, is a closed bounded convex set in E_n. Moreover, if H_0 and H_1 are respectively supporting closed half-spaces of B_0 and B_1 such that H_1 is a translate of H_0 then the translate H_t of H_0 which is a supporting half-space of B_t may be written $H_t = (1-t)H_0 + tH_1$. If x_0 is a point of B_0 in the boundary of H_0 and x_1 is a point of B_1 in the boundary of H_1 then $x_t = (1-t)x_0 + tx_1$ is a point of B_t in the boundary of H_t.

To apply this to the problem, let B_0 be the convex hull of C_1 and B_1 be the convex hull of C_2. Then with $t = \frac{1}{2}$ we have that $B_{1/2}$ is the convex hull of C and C lies in its boundary; *i.e.*, C is convex. Now $x_0 \in B_0$ and $x_1 \in B_1$ implies that $\frac{1}{2}(x_0 + x_1) \in B_{1/2}$ and hence no such point lies below C. Finally, place B_0 and B_1 in parallel planes in E_3. Then the union of the planar sets B_t, $0 \leqq t \leqq 1$, gives a convex solid and part of the surface of this solid is that required by part (c). If

the length of C_1 is L_1 and the length of C_2 is L_2, then the length of C is $\frac{1}{2}(L_1+L_2)$. It is not necessary that the curves have continuously turning tangents to establish these results.

Also solved by the proposers.

A Functional Equation

4811 [1958, 712]. *Proposed by Leonard Carlitz, Duke University*

Find the most general solution of the functional equation

$$2^{n/2}f_n(x\sqrt{2}) = \sum_{r=0}^{n}\binom{n}{r}f_r(x)f_{n-r}(x) \tag{1}$$

that is analytic at the origin.

Solution by A. E. Danese, Union College, Schenectady, N. Y. Let

$$g(x, y) = \sum_{n=0}^{\infty}\frac{f_n(x)y^n}{n!}, \qquad |y| < 1. \tag{2}$$

Then, there exists a region about the origin throughout which $g(x, y)$ is continuous. A simple transformation yields

$$g(\sqrt{2}x, \sqrt{2}y) = \sum_{n=0}^{\infty}\frac{2^{n/2}f_n(x\sqrt{2})y^n}{n!}$$

which, in view of (1), becomes $g(\sqrt{2}x, \sqrt{2}y) = g^2(x, y)$. The general continuous function which satisfies this functional equation is

$$g(x, y) = \exp(ax^2 + 2bxy - c^2y^2),$$

where a, b, c are arbitrary constants. From (2) we obtain

$$\exp(2bxy - c^2y^2) = \sum_{n=0}^{\infty}\frac{e^{-ax^2}f_n(x)y^n}{n!}.$$

Expanding the left side and equating the coefficients of y^n we obtain

$$f_n(x) = c^n e^{ax^2}H_n(bx/c),$$

where

$$H_n(x) = n!\sum_{i=0}^{[n/2]}\frac{(-1)^i(2x)^{n-2i}}{i!(n-2i)!}$$

are the nth degree Hermite polynomials defined by

$$e^{2xy-y^2} = \sum_{n=0}^{\infty}y^nH_n(x)/n!.$$

By analytic continuation, $f_n(x)$ is the solution of the functional equation for all x.

Also solved by J. H. Hodges, Richard Otter and Robert Weinstock, W. F. Trench, and the proposer.

Necessary and Sufficient Condition for a Polynomial

4813 [1958, 712]. *Proposed by H. D. Brunk, University of Missouri*

Given a differentiable function $f(x)$ such that to each $x \in (0, 1)$ there corresponds a positive integer $k = k(x)$ for which $f^{(n)}(x) = 0$ for all $n \geqq k$. Prove f is a polynomial.

Solution by R. P. Boas, Jr., Northwestern University. More generally, it can be proved that if for each x there is a $k = k(x)$ such that $f^{(k)}(x) = 0$, then f is a polynomial. (Corominas and Sunyer Balaguer, *Revista Mat. Hisp.-Amer.* (4) 14 (1954), 26–43; *Math. Reviews* 15, 942.) A proof is sketched in the review. Since the original paper is not readily accessible a (slightly different) proof is given here.

Let E_n be the set of points x for which $f^{(n)}(x) = 0$. Every x is in at least one E_n. By Baire's theorem there is a subinterval I in which some E_n is everywhere dense. Since $f^{(n)}$ is continuous, $f^{(n)}(x) = 0$ throughout I and f coincides in I with some polynomial. If I is not all of $(0, 1)$, repeat the reasoning with any remaining part of $(0, 1)$, and so on. We thus obtain a dense open set in each of whose component intervals f coincides with some polynomial. The complement H of this set is closed: we next show that, if not empty, it is perfect. If H is not perfect it has an isolated point, which is the common endpoint of two intervals on each of which f coincides with a polynomial. If n exceeds the degree of both polynomials, $f^{(n)}(x) = 0$ for x in both intervals, so f coincides with some polynomial in the union of the two intervals, and at their common endpoint by continuity; so the point cannot belong to H after all.

Now, since H is perfect, if it is not empty we can consider it as a complete metric space and apply Baire's theorem to it. Some E_n is then dense in some neighborhood in H, that is in the part of H that is in some interval J. In other words, there is an interval J that contains points x of H with $f^{(n)}(x) = 0$ for every such x (the same n for all x). J also contains intervals K complementary to H (since H is nowhere dense), and in each K, $f^{(m)}(x) = 0$ for some m. If $m \leqq n$, $f^{(n)}(x) = 0$ in K by differentiating. If $m > n$, we have $f^{(n)}(x) = f^{(n+1)}(x) = \cdots$ at the endpoints of K, by differentiating over H (since these endpoints are points of H). Then by integrating $f^{(m)}$ repeatedly we get $f^{(n)}(x) = 0$ throughout K. The same reasoning applies to every K, so $f^{(n)}(x) = 0$ throughout J. Thus J contains no points of H after all. This contradiction means that H was empty to begin with, so there was only one interval I and f coincides with a polynomial throughout $(0, 1)$.

Also solved by Robert Breusch and A. B. Willcox, J. M. Horváth, A. F. Kaupe, Jr., James Misho, and Henry Helson and the proposer.

The Greatest Integer Function

4814 [1958, 713]. *Proposed by D. J. Newman, AVCO Research, Wilmington, Mass.*

Given that $[n\alpha]+[n\beta]=[n(\alpha+\beta)]$ for all positive integers n. Prove that one of α, β is an integer.

Solution by Leonard Gewirtzman, University of Pennsylvania. Removing the integer parts of α, β in the given equation, we need consider only $0\leqq\alpha<1$, $0\leqq\beta<1$. We have $\alpha+\beta<1$, otherwise the hypothesis does not hold for $n=1$. Write $(x)'$ for $x-[x]$, and assume $\alpha>0$ and $\beta>0$. Then the hypothesis implies

$$(n(\alpha+\beta))' = (n\alpha)' + (n\beta)'. \tag{1}$$

Corresponding to any $\epsilon>0$ there exists an integer m such that

$$(m(\alpha+\beta))' < \epsilon, \tag{2}$$

since if $(\alpha+\beta)=p/q$, rational, we can take $m=q$; while if $(\alpha+\beta)$ is irrational, it is well known that the set $\{(m(\alpha+\beta))'\}$, $m=1, 2, \cdots$, is everywhere dense on $(0, 1)$. Therefore (1) implies $(m\alpha)'<\epsilon$, $(m\beta)'<\epsilon$. Choose $\epsilon<\min(\alpha, \beta)$. Then

$$\begin{aligned} [(m-1)(\alpha+\beta)] &= [m(\alpha+\beta)] - 1, \\ [(m-1)\alpha] = [m\alpha] - 1, &\qquad [(m-1)\beta] = [m\beta] - 1. \end{aligned} \tag{3}$$

By the original hypothesis, $[(m-1)(\alpha+\beta)]=[(m-1)\alpha]+[(m-1)\beta]$, whence by (3) we obtain $[m(\alpha+\beta)]=[m\alpha]+[m\beta]-1$, which contradicts the same hypothesis. Therefore $\alpha>0$ and $\beta>0$ is impossible and the theorem follows.

Also solved by A. C. Aitken, J. H. Hodges, D. C. B. Marsh, M. Sugunamma, and the proposer.

Editorial Note. Hodges bases his solution on a similar result of Jacobsthal. See *Norske Vid. Selsk, Forh.*, Trondheim, 30 (1957). 1–5.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Numerical Analysis (2nd ed.). By D. R. Hartree. Oxford University Press, London, 1958. $6.75.

As the first edition (1952) of Professor Hartree's book is firmly established on the shelf of all serious numerical analysts, we need only, in reviewing its second edition, to note some of the changes. There has been added a fuller treat-

ment of the Gaussian quadrature formula and of the solution of ordinary differential equations with two point boundary conditions. The material on hyperbolic differential equations has been slightly augmented. Three pages have been added on Whittaker's cardinal function—a theory which certainly deserves wider knowledge. The orientation of the book is toward the computer who uses a desk machine and indeed some of the material on programming which appeared in the first edition has been withdrawn. Still, it gives one a slight jar to read on page 11 that the main tools for numerical work are desk machine, tables, slide rule, and graph paper.

PHILIP J. DAVIS
National Bureau of Standards

Modern Mathematics for the Engineer. Edited by Edwin F. Beckenbach. McGraw-Hill, New York, 1956. xx+514 pp. $7.50.

It seems impossible for one man to criticize a collection of surveys such as this, written by a panel which is extremely well qualified. The problem is to ascertain a utility to the reader of the review and attempt to optimize it within these constraints.

It is stated in the preface that the committee sought to institute a series of lectures that would generate in the minds of engineers and applied scientists engaged in research, design, and administration, an awareness of the recent rapid advancement in applied mathematical thought—an advancement made possible in part by recent advances in basic mathematics and statistics and by the development of the analogue devices and digital computing machines of extremely high capacity and speed.

A prospective reader of the book should be forewarned of certain expectations. He should not expect a handbook of how to solve problems, nor even a coverage of the applicable cases, although such an undertaking should make a most profitable follow-up. He will find presented in each topic an excellent "feeling" for solving problems in the area, and when he should apply these "tools," provided he has more than a modicum of mathematical experience with the topic beforehand. With the exception of Wiener's article, which presupposes a maturity of thought concerning integration (working in the field requires a knowledge of Lebesgue integration, but gleaning the essence of the article requires only the realization that the Lebesgue integral has the desired properties), most of the articles can be appreciated with a background of ordinary differential equations and matrix theory (which is presented in the book).

The reader will not find applications of the analysis and design of experiments, regression and multivariate analysis to engineering problems nor analysis of error in interpolation problems, although these, too, might provide another volume. It seems to the reviewer that the book should prove most profitable to graduate students in physics, engineering, or applied mathematics, to industrial mathematicians or mathematical engineers, to industrial research teams,

and to administrators of both industrial and academic organizations who will consult with professional people in these fields in order to ascertain the trends in the applications of mathematics.

Part One has seven chapters: linear and nonlinear oscillations, by S. Lefschetz; stability theory, by R. Bellman; exterior ballistics, by J. W. Green; calculus of variations, by M. R. Hestenes; hyperbolic equations, by R. Courant; elliptic equations, by M. M. Schiffer; elastostatics, by I. S. Sokolnikoff. Part Two has five chapters: prediction, by N. Wiener; games, by F. Bohnenblust; operations research, by G. W. King; dynamic programming, by R. Bellman; Monte Carlo methods, by G. W. Brown. Part Three has seven chapters: matrices, by L. A. Pipes; functional transformations, by J. L. Barnes; conformal mapping, by E. S. Beckenbach; nonlinear methods, by C. B. Morrey, Jr.; relaxation methods, by G. E. Forsythe; methods of steep descent, by C. B. Tompkins; high-speed computing devices, by D. H. Lehmer. Each chapter was originally given as a lecture in an extension course at the University of California, Los Angeles, and also at the Corona Laboratories of the National Bureau of Standards.

R. B. Deal
Oklahoma State University

Eigenfunction expansions associated with second-order differential equations, Part II. By E. C. Titchmarsh. Clarendon Press, Oxford, 1958. xi+404 pages.

This scholarly monograph is a sequel to Part I, published in 1946. The author states that "the whole work is the result of an attempt by an 'analyst' to understand those parts of quantum mechanics which can be regarded as exercises in analysis." The object of study is the partial differential equation $\Delta\psi+(\lambda-q(\mathbf{x}))\psi=0$, where Δ is Laplace's operator, x is an n-dimensional real variable, and q is a given function of $\mathbf{x}$. Expansions in a rectangle, in the whole plane, and in more than two dimensions are treated. Properties of the spectrum are studied, distribution of the eigenvalues for certain classes of $q(\mathbf{x})$ is determined, convergence questions are taken up, perturbation theory is described, and the special case of periodic $q(\mathbf{x})$ is dealt with. The last chapter is, so to say, a mathematical appendix, where many needed theorems are collected.

As the foregoing summary indicates, this is a book for experts or those who wish to become experts. Most of the results have appeared in the periodical literature, but they are here collected for the first time. Professor Titchmarsh's style is as usual very clear. He starts with simple cases and with formalities, as he calls them, which are as a rule plausibility arguments. Following these, the reader finds himself very rapidly drawn into computations almost grotesque in their complexity. In his preface, Professor Titchmarsh writes: "It seems that physicists do not object to rigorous proofs provided that they are rather short and simple . . . . Unfortunately, it has not always been possible to provide proofs of this kind." With the last sentence, the reviewer is in complete agreement.

Edwin Hewitt
University of Washington

Physics and Mathematics, Ser. I, Vol. II. *Progress in Nuclear Energy*. Edited by D. J. Hughes, et al. Pergamon Press, New York, 1958. vii+375 pp. $14.00.

This book, like its predecessor (Volume I), aims to provide a review of recent progress in the experimental and theoretical aspects of nuclear reactor physics. Of the eight chapters, most are concerned with organizing the mass of relevant empirical data for anyone interested in making reactors. However, the eighth chapter on Monte Carlo Methods in Transport Problems by G. Goertzel and M. H. Kalos is a self-contained treatment of its subject matter which is of considerable interest to applied mathematicians. This chapter provides a compact introduction to Monte Carlo Methods, their relation to game theory and digital computer methods, and well-chosen examples of how they can be applied to obtain approximate solutions for the integral equations and partial differential equations which arise in reactor physics.

DAVID L. FALKOFF
Brandeis University

Mathematics for the Layman. By T. H. Ward Hill. Philosophical Library, New York, 1958. 339 pp. $4.75.

This is a book on what G. H. Hardy called "school" mathematics. It covers roughly the same material as Hogben's *Mathematics for the Million*. The main emphasis is on arithmetic, elementary algebra, and practical plane geometry, with briefer treatments of trigonometry, analytic geometry, and the calculus. Thus it does nothing to convey to the reader the nature of the mathematics of today. One must, however, grant an author the privilege of writing on whatever topic he pleases, and, after lamenting the fact that the book is not called "A Manual of Practical Old-Fashioned Mathematics for the Layman," pass on to its intrinsic merits and defects.

By and large, the book is clearly written with excellent explanations of the procedures used in arithmetic and elementary algebra. The author does have a talent for simplification and popularization. Naturally, he stresses examples and applications and there is little to indicate that one of the most important tasks of the mathematician is to provide rigorous proofs. In fact, in one of the rare occasions when a proof is given, it is incorrect. Thus the author (p. 27) "proves" the infinitude of primes as follows: "Multiply together all the numbers in the 'run' from 1 to (say) 10. Add 1 to the product. The result must be a prime number." And on page 125 we read that " . . . an important principle of mathematics . . . . It is that as all the 'plotted' points (and we can easily increase the number) lie upon a straight line," while on page 132 we have "The old definition [of a point] says 'a point has position but no magnitude.' It is a perfectly good definition, but rather top-heavy for such a simple idea. For a point is just a mark used to fix a particular position."

Other errors noted were in the definition of a circle (p. 139); the use of c/m in discussing $y=mx+c$ without specifying $m\neq 0$ (p. 247); the writing of $\int x^n dx$

$=(x^{n+1})/(n+1)+c$ with no restriction on n (p. 313); and the statement that "$\cdots x^3+y^3+2xy=1$ where it is not possible to calculate y directly in terms of x" (p. 257).

The sale of such books on "school" mathematics indicates the widespread desire of people to become mathematically literate. But, just as no "popular" history should be excused errors in dates just because it is for the layman, one cannot excuse gross mathematical errors in "popular" mathematics. And, while we may only lament that modern mathematics is not even touched upon, we can actively protest the attitude that elementary mathematics is just a collection of rules—no matter how clever.

ROY DUBISCH
Fresno State College

Mathematical Tables and Formulae (6th Ed.). By F. J. Camm. Philosophical Library, New York, 1958. 144 pp. $2.75.

C.R.C. Standard Mathematical Tables (11th Ed.). Chemical Rubber Publishing Company, Cleveland, 1958. 480 pp. $3.00.

Handbook of Calculus, Difference and Differential Equations. By E. S. Cogan and R. Z. Norman. Prentice-Hall, Englewood Cliffs, N. J., 1958. 263 pp. $6.00.

The first volume under review costs almost as much as the student edition of the Chemical Rubber Publishing Company's *Handbook of Physics and Chemistry* and contains about one-hundredth as much material.

The C.R.C. *Standard Mathematical Tables* are so well known that little need be said about this new edition. It has been enlarged here and there and a section on solutions of ordinary and partial differential equations has been added. It is still about the best table of its kind.

The Cogan and Norman volume contains the usual logarithmic, trigonometric, exponential and hyperbolic tables. It also contains a small table of integrals and two chapters are devoted to solutions of ordinary differential equations. A unique feature is the table of solutions of difference equations. These are well put together and should be helpful to users of this handbook.

The explanatory material, however, shows signs of having been prepared in haste. The inverse of the derivative operator D is an operator, not an indefinite integral. The set of all functions which satisfy a given functional equation is called the *general solution* of the equation (p. 12). This definition leads to error and confusion and it is small consolation to find (p. 229) that the authors did not mean it after all.

When one is talking about a function f whose values are given by $f(x)=Cx^2$, it seems undesirable to write $f=Cx^2$. Upper graph, p. 153, has a marking which should read $p>0$ instead of $p=0$. Upper graph, page 155, is incorrectly placed with respect to the y-axis. Graph (b), page 156, the three graphs on page 157 and/or the explanatory material are in error. Formula (15), page 177, is

wrong. The general form of the nonlinear equation of the first degree is *not* $F(x, f, Df) = 0$ (p. 229).

The integral formula (50), page 189, is wrong but here the authors are in good company: Many calculus books and tables of integrals write

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \cos^{-1}\left(\frac{a}{x}\right).$$

This is false; the correct answer is $(1/a)\sec^{-1}(x/a)$. The error is elementary:

$$D\cos^{-1}\left(\frac{1}{x}\right) = \frac{1}{x^2\sqrt{\dfrac{x^2 - 1}{x^2}}} \neq \frac{1}{x\sqrt{x^2 - 1}}.$$

It is remarkable that this basic error, repeated in many similar integrals, should still exist today.

No wonder we have not yet reached the moon. Maybe our missilemen are using integral 21, 125, 128, 142, 147, 151, or 154 (each wrong) in C.R.C. *Standard Mathematical Tables.*

C. O. Oakley
Haverford College

College Algebra. (4th ed.) By J. B. Rosenbach, E. A. Whitman, B. E. Meserve and P. W. Whitman. Ginn, New York, 1958. xiv+579+xlvi pp. $5.25.

This seems a particularly good book on college algebra. In the Preface is found a clear statement as to how the fourth edition differs from earlier ones.

Especially to be commended are the following: historical notes, with dates given for mathematicians, etc.; warnings, in particular that on avoiding the use of a necessary condition to prove convergence (p. 542); work with complex numbers; footnotes with references to other books for further material; treatment of one-to-one correspondence, absolute value, inequalities, division by zero; a very large number of exercises; number systems with bases other than 10; the "fundamental assumptions" of algebra; problems leading up to more advanced mathematics; "measure of accuracy" of a number; problems from the Rhind Papyrus and from a Greek anthology (c. 500 A.D.); miscellaneous exercises at the ends of many chapters; the illustration of mathematical induction, using a row of dominoes; exercise 1 on p. 552; the statement of the ratio test, and the warning (p. 550) about its use.

A very few errors have been noted. On page 83, Example 2 (above exercises), numerator after first equal sign, write $9^{1/2}$, not 9^1. On page 94, Illustration 1, last sentence, write $\sqrt{-b}$ for $\sqrt{-3}$. On page 311, last line write $2k\pi$ for $2k_\pi$. On page 542, exercise 8, second term should be 4/9, *if nth* term formula holds for all terms. The form of the "check" on page 488 seems so much better than that found earlier in the book (p. 154 for example).

Statistics, matrices and infinite series could be omitted. The first does not seem to be college algebra. The second will leave some students feeling that they know more about the subject than they do. The third could be left for a more advanced text, though it is excellently treated here.

Students of this book should not get a dozen points, instead of three, for the graph of a straight line. They should end the course with an excellent idea of what a general proof is. And they should certainly not be afraid of "story problems."

MARION E. STARK
Wellesley College

Mathematics and Logic for Digital Devices. By James T. Culbertson. Van Nostrand, Princeton, N. J., 1958. x+224 pp.

This book contains a collection of elementary mathematical and logical topics which are applicable to the study of digital devices.

The central notion used is that of the nerve net (systems of receptors, central neurons, and effectors) as an exemplification of a digital device. To the reviewer this is a mistake since the necessary mathematics can be discussed with reference to much less controversial models now in wide use such as Von Neumann diagrams or, more directly, relay or electronic circuit schemata.

The topics discussed include elementary number theory, permutations and combinations and elementary probability theory, conversion of number system bases, traditional logic, algebra of classes, and 2-valued Boolean algebra.

The book is very clearly written. It would probably be most useful as a text for nontechnical students who have had a freshman mathematical-concepts type course. Also, it would probably be more useful if it contained a brief discussion of computers and computing circuits, if only to motivate the book. The meager assortment of examples of switching circuits in the last chapter is not likely to whet the student's appetite for a study of the deeper and more interesting parts of computational mathematics.

RAYMOND J. NELSON
Case Institute of Technology

Foundations of Information Theory. By Amiel Feinstein. McGraw-Hill, New York, 1958. x+135 pp. $6.50.

From the author's preface: "The intention of the author in writing this book is to provide a concise but rigorous exposition of the fundamentals of the mathematical theory of information. Since the basic work of C. E. Shannon appeared in 1948, a good deal of work has been done in this field. Nevertheless there exists at the present time (January 1957) no single reference to which one interested in this subject can turn for a presentation which is up to date and yet reasonably complete." A. I. Khinchin in one of his fundamental papers on information theory written in 1956 expressed the same opinion.

In a sense the book is an outgrowth of the author's research papers on information theory. However, if one looks for "entropy" in this book he will find it mentioned only in the titles of some of the references. The author apparently does not like to use this name borrowed from statistical mechanics for the function, $H = -\sum p_i \log p_i$.

The book will be of interest to people trained both in mathematics and in communication engineering. Although the mathematician will appreciate its high level of rigor, the nonmathematical engineer will be lost in attempting to use the book.

ROBERT E. GREENWOOD
The University of Texas and
General Analysis Corporation

Introduction to Difference Equations. By Samuel Goldberg. Wiley, New York, 1958. xii+260 pp. $6.75.

Although intended primarily for social scientists, this elegant volume on difference equations will be of great value to any student of the subject. The reader need only have facility with elementary techniques of algebra and trigonometry; other mathematical results necessary for particular topics are included in the text in a very concise and intuitive fashion. The overall impression of the book is one of completeness and rigor, excellent motivation, and ease of reading.

The introduction and first three chapters treat the calculus of finite differences, the operator notation, natural problems leading to difference equations, and a complete discussion of the general nth order linear difference equation. Many important properties of the solutions of these equations are developed and applied to problems of considerable interest, including examples from economics, psychology, sociology and probability theory. The fourth and final chapter contains a number of special topics, including stability, the notion of generating functions, characteristic-value problems and matrix methods. At the same time the analogy between difference and differential equations is illustrated. This chapter also serves the purpose of drawing the whole subject of difference equations together with other, perhaps better-known techniques, making it properly a part of the whole, rather than an isolated topic.

Each chapter has starred sections requiring slightly more advanced knowledge, which may be omitted by the beginner as they are not referred to elsewhere in the text. The careful preparation, logical motivation, and easy rigor of this book make it ideal for use in teaching a first course in difference equations at, say, the junior level.

GORDON LATTA
Stanford University

BRIEF MENTION

They Come for the Best of Reasons—College Students Today. By W. Max Wise. American Council on Education, Washington, D. C., 1958. xi+65 pp. $1.00.

This inexpensive publication of the American Council on Education should provoke debate and inquiry on your own campus concerning the nature of the students whom you are serving.

The Mathematics Student. W. W. Sawyer, Ed. National Council of Teachers of Mathematics, 1201 Sixteenth St. N.W., Washington, D. C., 1958. 8 pp. $0.30 per year (4 issues).

A journal for grades 7 through 12. Sold only in groups of five or more subscriptions to a single address. This excellent little journal should certainly be called to the attention of high school and junior high school teachers throughout the country. Our congratulations to Mr. Sawyer, the editor, on the enlarged format and the addition of color.

An Emerging Program of Secondary School Mathematics. By Max Beberman. Harvard University Press, Cambridge, Mass., 1958. 44 pp. $1.50.

This authoritative explanation of the Beberman project of the University of Illinois' Committee on School Mathematics (U.I.C.S.M.) was presented at Harvard University as the Inglis Lecture in 1958. It is certainly of considerable interest to mathematicians everywhere.

College Algebra, (2nd ed.). By Thurman S. Peterson. Harper, New York, 1958. viii+413 pp. $4.00.

A genuine revision of Peterson's earlier *College Algebra* text. In the opinion of this reviewer it pays considerable lip service to "modern mathematics" without capturing its spirit. The text is still presented in a "rule for this" and "rule for that" type of presentation, with very little explanation of *why* the rules hold.

The Algebra of Electronics. Solving Circuit Problems. By Chester H. Page. Van Nostrand, Princeton, N. J., 1958. x+258 pp. $8.75.

Not a mathematical book, but it certainly is a book which this reviewer would have enjoyed having while he was in high school. High school algebra and a touch of the very simplest ideas from calculus are used to obtain solutions to the basic problems of electronic circuits.

Fundamentals of Digital Computers. By Mathew Mandl. Prentice-Hall, New York, 1959. xi+297 pp. $5.00.

Your reviewer "learned" a number of remarkable "facts" from this little volume. For example, it states that "the (I.B.M.) 650 magnetic core memory provides storage of the 20,000 characters in such a manner that the information is available at 3,528,000 characters per minute. Each of the 20,000 memory positions can be contacted individually at any time. Hence they can be grouped to form fields of any size or records of any length."

Designing the Mathematics Classroom. By Lawrence P. Bartnick. National Council of Teachers of Mathematics, Washington, D. C., 1957. iii+40 pp. $1.00.

This publication of the National Council of Teachers of Mathematics may prove a helpful reference in these times of imminent building of classrooms. One wonders from the list of tools and equipment which are suggested as essential, desirable, or possible in

a mathematics classroom whether mathematics may be more of a hand-skill than a mental skill. Nevertheless, there are some thought-provoking ideas in the small booklet, and it is well worth the modest price.

Guide to the Literature of Mathematics and Physics, Including Related Works on Engineering Science. By Nathan Grier Parke III. Dover, New York, 1958. xviii+436 pp. $2.49.

The first 74 pages of this interesting revision of the author's 1947 work are devoted to a discussion of the why, how, and where of library research and study. Certainly this guide could be very handy for anyone wishing to browse a bit, or for intensive study of some particular field.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor R. B. Deal, Jr., Oklahoma State University, represented the Association at the inauguration of Dr. B. G. Henneke as President of the University of Tulsa on April 16, 1959.

Professor I. L. Battin, Trenton State College, represented the Association at the inauguration of M. W. Gross as President of Rutgers, the State University, on May 6, 1959.

Associate Professor Mabel S. Barnes, Occidental College, represented the Association at the inauguration of Dr. Ralph Prator as the first President of San Fernando Valley State College on May 7, 1959.

Professor Hobart Bushey, Hunter College, represented the Association at the dedication of the Bronx Community College and the inauguration of Dr. Morris Meister as President of the college on May 11, 1959.

Professors Samuel Eilenberg, Columbia University, and K. O. Friedrichs, New York University, have been elected to membership in the National Academy of Sciences.

Professor K. W. Wegner, Carleton College, has been awarded a Fulbright Educational Exchange Grant to teach at National Taiwan University, Taipei, Taiwan, during the 1959–60 academic year.

Arizona State University: Prof. J. E. Freund has resigned as Department Chairman and has returned to full-time teaching; Professor L. L. Lowenstein has been appointed Department Chairman; Dr. Max Dengler, AiResearch Manufacturing Company, has been appointed Professor; Assistant Professor W. O. Portmann, Case Institute of Technology, has been appointed Associate Professor; Assistant Professor N. W. Savage, Polytechnic Institute of Brooklyn, has been appointed Associate Professor; Dr. L. T. Smith, Sequoia Adult School, has been appointed Assistant Professor; Adm. James Cohn

(USN Ret.) has been appointed Assistant Professor; Mrs. Wilma Thompson, University of Wyoming, Mrs. Gladdis Loehr, Baylor University, Adm. C. C. Seabury (USN Ret.) and Mr. W. R. Neal have been appointed Instructors; Instructors Albert Romano and George Peck have been promoted to Assistant Professors.

University of British Columbia: Associate Professor B. N. Moyls has been promoted to Professor; Assistant Professor Marvin Marcus has been promoted to Associate Professor; Dr. Fred Brauer, Dr. Charlotte Froese, Mr. Henryk Minc and Dr. C. A. Swanson have been promoted to Assistant Professors; Dr. N. J. Divinsky, University of Manitoba, has been appointed Associate Professor; Dr. D. W. Bressler, Tufts University, Mr. M. W. Katz, University of Illinois, and Dr. J. H. Lindsay, Jr., University of Toronto, have been appointed Instructors.

Brown University: Professor P. R. Masani, Institute of Science, Bombay, India, has accepted an appointment as Lecturer; Dr. D. J. Newman, Massachusetts Institute of Technology, has been appointed Assistant Professor; Associate Professor John Wermer has been given a leave of absence for the year 1959–60 to accept an appointment as a Visiting Lecturer at Harvard University; Mr. R. W. Rishel has been appointed Instructor for the year 1959–60.

Canisius College: Assistant Professor R. F. Tidd has been promoted to Associate Professor; Miss June McArtney, University of Buffalo, has been appointed Assistant Professor.

University of Chicago: Associate Professor A. P. Calderon, Massachusetts Institute of Technology, has been appointed Professor; Dr. K. R. Stromberg, Yale University, and Dr. L. E. Baum, National Science Foundation Fellow, have been appointed Research Lecturers; Dr. D. R. Hughes has been appointed ESSO Research Lecturer; Assistant Professors Eldon Dyer and R. K. Lashof have been promoted to Associate Professors.

Haverford College: Professor Hans Rademacher has been appointed Philips Visitor for 1959–60; Assistant Professor Louis Solomon, Bryn Mawr College, and Mr. James Brooks, University of Michigan, have been appointed Assistant Professors; Assistant David Harrison has been appointed Assistant Professor at the University of Pennsylvania; Assistant Professor R. J. Wisner has received a National Science Foundation Science Faculty Fellowship and will spend the year at the Institute for Advanced Study.

Knox College: Dr. J. R. Mayor, University of Wisconsin and AAAS, an alumnus of Knox, was given a Doctor of Laws degree at the June Commencement for his work in mathematical and scientific education; Professor Rothwell Stephens was made the first Distinguished Service Professor on the Robert J. Szold Foundation; Associate Professor A. O. Lindstrum, Jr. was promoted to Professor; Assistant Professor C. R. Ohman will be on leave to spend a year as Instructor at Princeton University; Mr. P. H. Yearout, University of Washington, has been appointed Assistant Professor.

Massachusetts Institute of Technology: Professor R. D. Schafer, University of Connecticut, has been appointed Professor and Deputy Head of the Mathematics Department; Associate Professor I. M. Singer has been promoted to Professor; Assistant Professors L. N. Howard and Hartley Rogers, Jr., have been promoted to Associate Professors; Dr. K. M. Hoffman, has been promoted to Assistant Professor.

University of Minnesota: Dr. A. M. Garsia, Massachusetts Institute of Technology, and Dr. C. D. Gorman, University of Washington, have been appointed Assistant Professors; Mr. J. T. Joichi, University of Illinois, Mr. F. B. Knight, Princeton University, Mr. C. L. Miracle, University of Kentucky, and Mr. D. A. Woodward have been appointed Instructors.

University of Nebraska: Professor Edwin Halfar has been appointed Acting Chairman of the Mathematics Department for 1959–60; Associate Professor L. K. Jackson has been promoted to Professor; Dr. G. H. Meisters, Duke University, has been appointed Assistant Professor; Professor Trevor Evans, Emory University, will be a

Visiting Professor; Professor C. C. Camp has retired with the title Professor Emeritus; Professor W. G. Leavitt will be on leave at Princeton University for the year 1959–60.

University of North Carolina: Dr. J. H. Wells has been promoted to Assistant Professor; Mr. G. W. Henderson, University of Texas, has been appointed Instructor.

Tulane University: Professor G. S. Young, University of Michigan, has been appointed Professor; Assistant Professors F. D. Quigley, Yale University, and Bernhard Banaschewski, Hamilton College, McMaster University, have been appointed Associate Professors.

University of Wisconsin, Mathematics Research Center, United States Army: The following have received appointments: Dr. Gaetano Fichera, University of Rome, Italy; Professor Haim Hanani, Israel Institute of Technology; Professor C. C. Hsuing, Lehigh University; Professor N. D. Kazarinoff, University of Michigan; Professor Cornelius Lanczos, Institute for Advanced Study, Dublin, Eire; Professor W. S. Loud, University of Minnesota; Professor A. M. Ostrowski, University of Basel, Switzerland.

Mr. D. S. Adorno, Harvard University, has accepted the position of Senior Research Mathematician at the Jet Propulsion Laboratory, California Institute of Technology.

Dr. D. O. Banks, Carnegie Institute of Technology, has been appointed Acting Assistant Professor at the University of California at Davis.

First Lt. D. R. Barr, University of Iowa, has been appointed Instructor at the U. S. Air Force Academy.

Mr. James Bercos, Lockheed Aircraft Corporation, Marietta, Georgia, has accepted the position of Research Mathematician with the RCA Service Company, Patrick Air Force Base, Florida.

Associate Professor Olivier Biberstein, Memorial University, St. John's, Newfoundland, Canada, has been appointed Professor Extraordinario at the University of Costa Rica.

Mr. J. R. Bienas, Indiana Technical College, has accepted the position of Engineer with the Philco Corporation, Philadelphia, Pennsylvania.

Professor Enrico Bompiani, University of Rome, Rome, Italy, has been appointed Andrew Mellon Professor of Mathematics at the University of Pittsburgh.

Mr. G. T. Boswell, White Sands Missile Range, has accepted the position of Engineer with Convair Astronautics, San Diego, California.

Dr. L. R. Bragg, Duke University, has been appointed Associate Professor at West Virginia University.

Mr. J. R. Brashear, R. A. Cummings, Jr., and Associates, Pittsburgh, Pennsylvania, has accepted the position of Data Processing Analyst with United States Steel, Pittsburgh, Pennsylvania.

Associate Professor Paul Brock, Purdue University and the University of Michigan, has accepted a position as Head, Industrial Dynamics Research, Hughes Aircraft Company, Culver City, California.

Captain E. E. Brown, Air Force Department, Stanford University, has been promoted to Major, United States Air Force, Offutt Air Force Base, Nebraska.

Mr. A. L. Buchman, Hutchinson-Central Technical High School, Buffalo, New York, has been appointed Associate in Mathematics with the State Education Department.

Assistant Professor W. O. Buschman, California State Polytechnic College, has been promoted to Associate Professor.

Mr. D. A. Celarier, Northwestern University, has accepted the position of Research Engineer "B" with Boeing Airplane Company, Pilotless Aircraft Division, Seattle, Washington.

Mr. D. R. Childs, Westinghouse Electric Corporation, Pittsburgh, Pennsylvania, has accepted the position of Staff Physicist with Allied Research Associates, Inc., Boston, Massachusetts.

Dr. Dorothy J. Christensen, Wellesley College, has been appointed Instructor at Reed College.

Mr. K. L. Conrad, Goodyear Aircraft Corporation, Akron, Ohio, has been promoted to Research and Development Engineer.

Assistant Professor E. H. Crisler, University of Notre Dame, has accepted the position of Principal Mathematician with the Advanced Development Laboratories, Bendix Aviation Corporation, South Bend, Indiana.

Dr. R. T. Dames, Massachusetts Institute of Technology, has accepted a position as a Member of the Technical Staff of Ramo-Wooldridge, Los Angeles, California.

Mr. L. D. Davis, University of Miami, has been appointed Part Time Instructor at the University of North Carolina.

Mr. August Deckert, Boeing Airplane Company, Renton, Washington, has accepted a position as Applied Science Representative with International Business Machines Corporation, South Gate, California.

Mr. F. H. Ditto, Northeastern University and Lincoln Laboratory, Massachusetts Institute of Technology, has accepted the position of Mathematician with the Systems Analysis Directorate, National Aviation Facility Experimental Center, Atlantic City, New Jersey.

Mr. F. A. Dunn, Jr., La Salle College, has been appointed Assistant Instructor at the University of Pennsylvania.

Mr. D. V. Easter, Sandia Corporation, Livermore, California, has accepted the position of Senior Engineer with the Martin Company, Orlando, Florida.

Mr. R. L. Edwards, Jr., Johns Hopkins University, has accepted the position of Engineer with Aircraft Armaments Inc., Cockeysville, Maryland.

Professor F. A. Ficken, University of Tennessee, has been appointed Professor at New York University.

Mr. M. V. Fiondella, University of Florida, has been appointed Instructor at the University of New Mexico.

Mr. L. C. Fletcher, Mt. Sterling High School, Mt. Sterling, Ohio, has been appointed Instructor at Eastern Kentucky State College.

Mr. J. B. Garner, Carleton College, has been appointed a Teacher for the American Board of Commissioner for Foreign Missions, Boston, Massachusetts.

Dr. H. H. Goldstine, International Business Machines Corporation, Yorktown Heights, New York, has been appointed Resident Manager of the Lamb Estate Research Center of the International Business Machines Corporation in the Town of Cortlandt, New York.

Associate Professor L. C. Graue, Coe College, has been appointed Associate Professor at Bowling Green State University.

Associate Professor Simon Green, University of South Carolina, has been appointed Associate Professor at the Assumption University of Windsor, Essex College, Windsor, Ontario, Canada.

Dr. Edison Greer, Lockheed Missiles and Space Division, Sunnyvale, California, has been appointed Professor and Head of the Mathematics Department at San Jose State College.

Dr. W. J. Hardell, Remington Rand Univac, St. Paul, Minnesota, has accepted the position of Engineer with the Radio Corporation of America, Moorestown, New Jersey.

Dr. D. L. Hartford, University of Kentucky, has been appointed Assistant Director of Testing Service, University of Kentucky

Mr. J. W. Haynes, Jr., Pearl Harbor Navy Shipyard, has accepted the position of Mathematical Physicist with the Dalmo Victor Company, California.

Mr. T. E. Hildick, Jr., Mississippi State College, has accepted the position of Engineer with the Western Electric Co., Winston-Salem, North Carolina.

Dr. T. R. Horton, International Business Machines Corporation, Washington, D. C., has been appointed Manager of 700–7000 Systems Marketing, Data Processing Division, White Plains, New York.

Mr. W. R. Hutcherson, Jr., United States Air Force, has accepted a position with the System Development Corporation, Santa Monica, California.

Professor S. B. Jackson, University of Maryland, will be on sabbatical leave during 1959–60 on an NSF Fellowship at the University of Washington

Mr. A. J. Janis, De La Salle Institute, Chicago, has been appointed Teacher at St. George High School, Evanston, Illinois.

Dr. M. P. Jarnagin, Jr., Naval Aviation Ordnance Test Station, Chincoteague, Virginia, has accepted a position as Mathematician at the Naval Proving Ground, Dahlgren, Virginia.

Mr. Raymond Kassler, Atlantic Design Company, Newark, New Jersey, has accepted the position of Analysis Engineer with the Data Processing Company, Elmhurst, New York.

Mr. W. M. Lindstrom, University of Kansas, has accepted the position of Applied Science Representative for International Business Machines Corporation, Hammond, Indiana.

Mr. J. M. Lowerre, University of Buffalo, has been promoted to Instructor.

Mr. R. D. Mason, Jr., American University, has accepted the position of Programmer with the Burroughs Corporation, Paoli, Pennsylvania.

Assistant Professor I. A. McCollum, North Carolina College at Durham, will study at the University of North Carolina on an NSF Faculty Fellowship during the year 1959–60.

Professor Norman Miller, Queen's University, Kingston, Ontario, Canada, has retired from active teaching.

Mr. D. E. Morrill, Western Electric Company, Winston-Salem, North Carolina, has accepted the position of Supervisor, Computer Applications Unit, Reaction Motors Division, Thiokol Chemical Corporation, Danville, New Jersey.

Mr. W. L. Morse, State College of Washington, has been aoppinted Instructor at Portland State College.

Assistant Professor W. O. J. Moser, University of Saskatchewan, Saskatoon, Saskatchewan, Canada has been appointed Associate Professor at the University of Manitoba, Winnipeg, Manitoba, Canada.

Dr. Z. I. Mosesson, The Prudential Insurance Company of America, Newark, New Jersey, has been promoted to Assistant Actuary.

Mr. A. M. Nagy, on sabbatical leave from the Pomfret School, Pomfret, Connecticut, has been appointed Assistant Professor at the University of Hawaii for 1959–60.

Mr. J. H. Ortman, Army Michigan Ordnance Missile Plant, Detroit, Michigan, has accepted the position of Senior Research Engineer for North American Aviation, Inc., Los Angeles, California.

Dr. Mary H. Payne, Inertial Systems Research Laboratory, Wyandanch, New York, has accepted the position of Principal Mathematician with Republic Aviation Corporation, Farmingdale, New York.

Mr. D. J. Persico, Ohio State University Research Foundation, has accepted a position as Associate Engineer in the Mathematical Analysis Section of Lockheed Aircraft Company, Burbank, California.

Associate Professor G. M. Petersen, University of New Mexico, has been appointed Lecturer at the University College of Swansea, Swansea, Great Britain.

Dr. W. N. Prentice, Denison University, has been promoted to Assistant Professor.

Assistant Professor John Raleigh, Lehigh University, has been appointed Assistant Professor at Temple University.

Professor O. J. Ramler, Catholic University of America, has retired with the title Professor Emeritus after 44 years of service, the last 8 years as Head of the Mathematics Department.

Mr. M. F. Reese, South Texas College, has accepted the position of Electronics Engineer with the Secode Corporation, San Francisco, California.

Dr. B. L. Reinhart, University of Michigan, has been appointed Assistant Professor at the University of Maryland.

Assistant Professor B. E. Rhoades, Lafayette College, has been promoted to Associate Professor.

Mr. Norman Schaumberger, The Cooper Union School of Engineering, has been appointed Assistant Professor at Bronx Community College.

Mr. T. A. Schoen, University of Cincinnati, has been appointed Instructor at the University of Dayton.

Professor J. P. Scholz, New Mexico Institute of Mining and Technology, has been appointed Professor and Head of the Department of Mathematics and Physics at the Western College for Women.

Visiting Associate Professor Seymour Schuster, Carleton College, has accepted the permanent position of Associate Professor.

Mr. R. A. Sebastian, Ballistic Research Laboratories, Aberdeen, Maryland, has accepted the position of Mathematician with Caywood-Schiller, Associates, Chicago, Illinois.

Mr. G. H. Silberberg, Bell Aircraft Corporation, Buffalo, New York, has accepted the position of Associate Engineer—Mathematical Analysis, with the Lockheed Aircraft Corporation, Burbank, California.

Mr. A. J. Silverman, Naval Ordnance Test Station, China Lake, California, has been appointed as a member of the Technical Staff, Space Technology Laboratories, Inc., Patrick Air Force Base, Florida.

Mr. D. J. Sine, Aero-Jet General, Frederick, Maryland, has accepted a position as Research Specialist for the Allegany Ballistics Laboratory, Cumberland, Maryland.

Sister Rose Marian, O.S.F., Ladycliff College, has been promoted to Assistant Professor and Chairman of the Division of the Natural Sciences.

Mr. D. J. Smith, Breeze Corporation, Inc., Newark, New Jersey, has accepted the position of Electronics Engineer for the Convair (San Diego) Division of the General Dynamics Corporation.

Mr. Irwin Stoner, Arma Division, American Bosch Arma Corporation, Garden City, New York, has accepted a position as Senior Mathematician with the Service Bureau Corporation, New York.

Associate Professor D. D. Strebe, University of South Carolina, has been promoted to Professor.

Mr. B. K. Swartz, Research Assistant, Los Alamos Scientific Laboratory, has been promoted to Staff Member.

Mr. Melvin Tainiter, American Bosch Arma Corporation, Garden City, New York, has accepted the position of Systems Engineer for International Electric Corporation, Paramus, New Jersey.

Assistant Professor Selmo Tauber, University of Kansas, has been appointed Associate Professor at Portland State College.

Mr. D. F. Templeton, Jr., University of Maryland, has accepted a position as Mathematician at the David Taylor Model Basin, Washington, D. C.

Miss Eileen J. Theisen, North American Aviation, Canoga Park, California, has accepted the position of Applied Mathematician with the Bell & Howell Company, Chicago, Illinois.

Mr. P. H. Thrower, International Business Machines Corporation, Dallas, Texas,

has been appointed Education Center Manager, IBM Corporation, Seattle, Washington.

Assistant Professor D. B. Tillotson, Northwest Nazarene College, is at the University of Kansas during 1959–60 on a NSF Science Faculty Fellowship.

Mr. R. A. Wonderly, Univac, Remington Rand Division, Sperry Rand Corporation, St. Paul, Minnesota, has been appointed Research Assistant at the University of North Carolina.

Mr. B. A. Yale, Cornell Aeronautical Laboratories, Buffalo, New York, has accepted a position as Mathematician with the Bendix Pacific Division, Bendix Aviation Corporation, North Hollywood, California.

Dr. William Zlot, City College of New York, has been appointed Associate Professor at Paterson State College.

Dean E. A. Bailey, Lagrange College, died on May 5, 1959. He had been a member of the Association for twenty-five years.

Professor E. T. Browne, University of North Carolina, died on March 31, 1959. He had been a member of the Association for thirty-eight years.

Dr. James A. Bullard, Professor Emeritus, University of Vermont and State Agricultural College, died on April 10, 1959. He was a charter member of the Association.

Professor H. P. Evans, University of Wisconsin, died on June 2, 1959. He had been a member of the Association for thirty years and was a member of the Board of Governors.

Professor T. E. Gravatt, Pennsylvania State University, died on February 12, 1959. He was a charter member of the Association.

Professor G. W. Hess, Howard College, died on March 7, 1959. He was a charter member of the Association.

Dr. D. G. Humm, Director, Humm Personnel Consultants, Los Angeles, California, died on May 17, 1959. He had been a member of the Association for seven years.

Professor J. J. Knox, Dakota Wesleyan University, died on April 10, 1959. He had been a member of the Association for thirty-eight years.

Associate Professor Emeritus L. T. Moore, Brooklyn College, died on April 16, 1959. He had been a member of the Association for thirty-six years.

Mr. G. N. Robinson, New Bedford, Massachusetts, died on January 28, 1959. He had been a member of the Association for thirteen years.

Assistant Professor C. A. Rogers, Colorado State University, died on May 30, 1959. He had been a member of the Association for five years.

Professor Emeritus E. I. Yowell, University of Cincinnati, died on March 12, 1959. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 294 persons have been elected to membership by the Board of Governors on applications duly certified.

PAUL ABRAMSON, M.A. (Columbia) Grad. Asst., Syracuse University

WADE L. ALLEN, Ph.D. (Florida) Senior Research Scientist, Litton Industries.

NORMAN L. ALLING, Ph.D. (Columbia) Asst. Professor, Purdue University.

GEORGE R. ANDERSON, Ed.D. (Pennsylvania S.U.) Professor, State Teachers College, Millersville, Pennsylvania.

PHILIP H. ANDERSON, M.A. (Notre Dame) Part-time Instr., Aquinas College; Teacher, Muskegon Catholic High School, Michigan.

MONA RAE ARMSTRONG, Student Humboldt State College.

ALFONSO G. AZPEITIA, Ph.D. (Madrid) Asst. Professor, University of Massachusetts.

MRS. CAROLYN RUTH R. BAHOUS, M.A. (North Carolina) Asso. Professor, Lynchburg College.

MARJORIE J. BAKIRAKIS, A.M. (Radcliffe) Instr., Hood College.

ROBERT W. BASS, Ph.D. (Johns Hopkins) Member, Research Institute for Advanced Study.

LEWIS E. BATSON, M.S. (Louisiana S.U.) Lecturer, University of Texas.

JAN R. BAUER, B.S. (Kent S.U.) Grad. Student, Kent State University.

JAMES N. BAUSCH, Student, University of Oklahoma.

GEORGE L. BAYER, M.A. (Loyola, Illinois) Teacher, Chicago Board of Education, Illinois.

ARMOND W. BEAR, M.S. (Marquette) Chairman of Dept., Rufus King High School, Milwaukee, Wisconsin.

HOWARD F. BECKSFORT, Ph.D. (Syracuse) Asso. Professor, Carroll College.

MICHAEL E. BENNETT, Student, DeWitt Clinton High School, New York, New York.

GUY M. BENSON, M.S. (Kansas City) Project Mathematician, I.B.M. Corp.

GEORGE W. BEST, B.S. (Union) Instr., Phillips Academy.

JAMES R. BOEN, Grad. Asst., University of Illinois.

WILLIAM J. BONINI, M.S. (Wyoming) Instr., Idaho State College.

DAVID M. BOODMAN, Ph.D. (Pittsburgh) Operations Analyst, Massachusetts Institute of Technology.

PAUL E. BOUDREAU, M.S. (Rochester) Programmer, I.B.M. Corp.

DONALD W. BOWAN, Student, Millikin University.

PAUL J. BOWLBY, M.S. in Ed. (Indiana) Teacher, Bellflower Unified School District, California.

JAMES W. BRADLEY, B.S. (Loyola, Maryland) Mathematician, Ballistic Research Labs.

DAVID F. BRAUCH, Student, University of Wisconsin.

LAWRENCE M. BREED, Student, Stanford University.

MRS. LUCILE C. BRIGGS, Computer, Boeing Airplane Co.

MILTON BRIZEL, M.A. (Columbia) Teacher, Fallsburgh Central School, New York.

HARRY BRODINE, M.A. (Fordham) Instr., State University of New York, Albany

JAMES T. BROGAN, Student, Mississippi State University.

WILLIAM A. BROWN, A.B. (Bowdoin) Grad. Student, University of Maine.

THOMAS J. BRUGGEMAN, B.S. (Dayton) Instr., Xavier University, Ohio.

PETER H. BUCKLEY, Student, University of Alberta.

GEORGE T. BUCKWALTER, B.S. in Ed. (West Chester S.T.C.) Instr., Valley Forge Military Academy.

GLENN A. BURDICK, B.S. (Georgia Inst. Tech.) Grad. Student, Georgia Institute of Technology.

VICTOR J. BUZZANCA, B.S. (Queens) Reliability and Quality Control Engineer, Arma Corp.

GLENN D. CAREY, A.B. (Wilkes) Engineer, Western Electric Co.

KENNETH E. CARLSON, B.S. (Texas Christian) Instr., North Texas State College.

CAPT. GEORGE C. CARLSTEDT, M.S. (Purdue) Instr., Bradley University.

KENNETH S. CARMAN, M.A. (Tennessee) Asso. Professor, Kansas Wesleyan University.

JOHN H. CASE, Student, Whittier College.

C. RONALD CASSITY, Ph.D. (Illinois) Senior Mathematician, New Mexico Institute of Mining and Technology.

VLADIMIR B. CERVIN, Ph.D. (Prague) Research Psychologist, Imperial Oil Limited.

JOSEPH T. CHELL, B.A. (British Columbia) Teacher, Burnaby South High School, British Columbia.

WILLIAM G. CHINN, A.B. (California, Berkeley) Teacher, San Francisco Unified School District, California.

ROBERT E. CLARK, Ph.D. (North Carolina) Asst. Professor, Virginia Military Institute.

ELEANOR M. CLIFFORD, Student, Anna Maria College.

N. DONALD COHEN, Student, Los Angeles State College.

MIRIAM V. N. COLLINS, M.A. (Columbia) Teacher, Haverford Township Junior High School, Pennsylvania.

ANNABELLE T. COMFORT, Student, University of Oklahoma.

SAM S. COMO, Student, Cooper Union; Asst. Office Manager, Bandt, Inc.

ARTHUR H. COPELAND, JR., Ph.D. (M.I.T.) Asst. Professor, Purdue University.

ALBERT G. COX, Student, Agricultural and Mechanical College of Texas.

KEITH J. CRASWELL, A.S. (Olympic) Grad. Student, University of Washington.

GORDON R. DAHLSTROM, B.A. (Chicago) Lithographic Technician, Rand McNally & Co.

COLDWELL DANIEL, III, Ph.D. (Virginia) Chairman of Econ. Dept., Mississippi Southern College.

WILLIAM H. DANNACHER, M.A. (Villanova) Asst. Professor, Villanova University.

DONALD B. DAVIS, B.S. (Carnegie Inst. Tech.) Grad. Student, Stanford University.

FRANK DEAN, Student, University of Redlands.

ANTHONY J. DEGENNARO, M.Ed. (Pennsylvania S.U.) Instr., Firelands High School, Oberlin, Ohio.

MARION E. DENINNO, A.B. (Washington & Jefferson) Chairman of Dept., Charleroi High School, Pennsylvania.

EDOUARD J. DESAUTELS, Student, University of Manitoba.

MORTON DEUTSCH, Ph.D. (M.I.T.) M.T.S., Bell Telephone Labs.

WILLIAM D. DEVITT, Ed.M. (Harvard) Teacher, Salem Classical and High School, Massachusetts.

ROY A. DOBYNS, M.A. (Vanderbilt) Asst. Professor, McNeese State College.

GERALD C. DODDS, B.S. (St. Vincent) Mathematician, Haller, Raymond & Brown.

PETER DORATO, M.S. in E.E. (Columbia) Instr., Polytechnic Institute of Brooklyn.

BRYAN M. EAGLE, Student, Southwestern College.

ROY M. ELLEDGE, Student, University of California.

LOWELL EMERLING, M.A. (Michigan) Senior Scientist, Lockheed Aircraft Corp.

HOWARD B. EMERSON, Student, State College of Washington.

GARRET J. ETGEN, Student, College of William and Mary.

WILLIAM D. ETLING, B.S. in Ed. (Kent S.U.) Temporary Instr., Kent State University.

SARAH R. EVANGELISTA, B.S. (Temple) Instr., Temple University.

W. BUELL EVANS, PH.D. (Illinois) Asso. Professor, Georgia Institute of Technology.

ROBERT P. EVERETT, A.B. (Georgia) 180 Westview Drive, Athens, Georgia.

EDWARD L. FADER, B.S. (Wyoming) Grad. Student, University of Wyoming.

MRS. ETTA Z. FALCONER, M.S. (Wisconsin) Head of Dept., Okolona College.

BARBARA A. FLORES, Student, Baylor University.

GUY M. FORMICHELLA, B.S. (C.C.N.Y.) Electronic Scientist, Rome Air Development Center.

MAURICE A. FORTIER, Clerk, National Defence, Ottawa, Ontario.

ALFRED P. FOX, B.A. (Temple) Mathematician, Burroughs Research Center.

GEOFFREY T. FOX, Student, University of Santa Clara.

PAUL O. FREDERICKSON, B.A. (Maryland) Grad. Student, University of Maryland; Asso. Mathematician, Applied Physics Lab., Johns Hopkins University.

MARVIN I. FREEDMAN, Student, Massachusetts Institute of Technology.

HAROLD G. FROEHLICH, M.A. (Fordham) Chairman of Dept., John Adams High School, Ozone Park, New York.

RICHARD E. FULLER, B.S. (Westminster, Pennsylvania) Head of Dept., Technical-Vocational High School, Hammond, Indiana.

SOPHRONIA A. GAMBLE, Student, William Smith College.

MRS. OLIVE R. GARDNER, M.A. (Minnesota) Teacher, Lyons Township High School and Junior College.

GEORGE C. GASTL, Student, University of Kansas.

PAUL C. GILMORE, Ph.D. (Amsterdam) Staff Mathematician, I.B.M. Research Centre.

ANTON GLASER, B.A. (Temple) Teacher, Upper Moreland High School, Willow Grove, Pennsylvania.

MRS. EVA W. GRAY, Ph.D. (Zurich) Instr., Douglass College.

JOSEPH GREENBAUM, M.S. (Columbia) Chairman of Dept., Long Beach High School, New York.

JOEL E. GREENBERG, Student, City College of New York.

MARY L. GRIGGS, M.A. (Georgia) Teacher, Piedmont College.

ROBERT L. GUDERJOHN, Student, Idaho State College.

PETER H. GUM, B.A. (Oberlin) Engineer, Melpar, Inc.

HADI HADDAD, M.A. (Texas) Grad. Student, University of Texas.
DOROTHEA C. HADSELL, M.A. (N.Y.S.C. for Teachers) Supervisor, Secondary Math., Albany, New York.
HAROLD W. HAGER, A.M. (Missouri) Instr., Southeast Missouri State College.
RAOUL HAILPERN, B.A. (London) Instr., University of Buffalo, Park School of Buffalo.
EDWARD M. HARRIS, B.S. (Colorado) Instr., San Francisco State College.
DAVID E. HARVEY, SR., Instr., Chicago Technical College.
FRANK HARWAYNE, B.A. (Brooklyn) Consulting Actuary, Frank Harwayne & Associates.
SIMON W. HECHT, Student, Trinity College.
KENNETH B. HENDERSON, Ph.D. (Ohio S.U.) Professor, University of Illinois.
HANS-HEINRICH W. HERDA, Student, Wayne State University.
MRS. GLORIA C. HEWITT, B.A. (Fisk) Grad. Student, University of Washington.
KENNETH M. HEWITT, Student, College of St. Thomas.
CHARLES HODES, A.M. (Columbia) Teacher, Polytechnic Institute of Brooklyn.
MRS. AIKO M. HORMANN, A.M. (Missouri) Mathematician, Westinghouse Research Labs.
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STANFORD H. JOHNSON, M.A. (Geo. Peabody) Asst. Professor, East Tennessee State College.
CONSTANCE J. JONES, M.A. (Texas) Instr., San Antonio College.
ROBERT L. JONES, A.B. (Hiram) Teacher, Chagrin Falls High School, Ohio.
WILLIAM E. JONES, A.M. (Chicago) Teacher, Belmont High School, Dayton, Ohio.
ROBERT V. KESTER, A.B. (Whittier) Acting Instr., University of Idaho.
ROBERT W. KING, M.A. (Mississippi Southern) Instr., Vanderbilt University.
CHARLES W. KINNEY, A.B. (Akron) Teacher, Buchtel High School, Akron, Ohio.
JOHN R. KINZER, Ph.D. (Geo. Peabody) Human Factors Scientist, System Development Corp.
V. D. KIRKLAND, M.A. (Georgia) The Citadel.
WILLIAM E. KIRWAN, II, Asst., University of Kentucky.
ISRAEL KLEINER, Student, McGill University.
BARRON G. KNECHTEL, M.A. (Northwestern) Instr., Orange Coast Junior College.
WILLIAM F. KOEHMSTEDT, Student, University of Santa Clara.
MRS. RUTH D. KOIDAN, B.S. (Geo. Washington) Grad. Student, George Washington University; Mathematician, NASA, Washington, D. C.
ROBERT W. KOLTS, B.A. (Utah) Teacher, West High School, Salt Lake City, Utah; Grad. Student, University of Chicago.
PAUL G. KUMPEL, JR., B.S. (Trenton S.C.) Instr., Lafayette College.
IRA A. KWEITKO, Student, Los Angeles State College.
FRANK LAMANTIA, M.S. (Omaha) Teacher, Thomas Jefferson High School, Council Bluffs, Iowa.
REV. JEAN-PAUL LANDRY, M.S. (Laval) Teacher, Seminaire de Saint-Jean, Quebec.
FRANCIS G. LASAK, M.S. (Michigan) Asst. Professor, New York City Community College of Applied Arts & Sciences.
CHARLES J. LAVIOS, Student, San Jose State College.
LEON LEBLANC, M.A. (Chicago) Asst. Professor, University of Montreal.
PIERRE LEDUC, B.S. (Montreal) Grad. Student, University of Montreal.
L. DAVID LEWIS, B.A. (Colorado) Teaching Asst., Stanford University.
RALPH D. LEWIS, B.S. (New York) Teacher, Malverne Jr. High School.
ERIC H. LINDBERG, B.S. (Gustavus Adolphus) Teacher, West High School, Minneapolis, Minnesota.
LEONARD J. LIPKIN, Student, Oberlin College.
BENJAMIN LIPSTEIN, Ph.D. (Columbia) Asso. Research Director, Benton & Bowles, Inc.
WILLIAM G. LISTER, Ph.D. (Yale) Professor, State University College on Long Island.
DONALD M. LONG, B.S. (Columbia) Teacher, Barnard School for Boys, Bronx, New York.
JAMES J. LOUGHNEY, M.S. (Bucknell) Applied Science Representative, I.B.M. Corp.
JAMES M. LOWERRE, B.S. (U. S. Military Acad.) Instructor, University of Buffalo.
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THOMAS H. MACGREGOR, A.M. (Pennsylvania) Instr., Rutgers University.
EDWIN S. MACK, B.S. (Ursinus) Grad. Student, Duke University.
TRAVIS E. MADOLE, Student, Texas A. & M.
KENNETH D. MAGILL, JR., B.S. (Shippensburg S.T.C.) Grad. Asst., Pennsylvania State University.
HARRY MALISOFF, Ph.D. (Columbia) Asso. Professor, Brooklyn College.
MITZI L. MALLARIAN, Student, North Dakota Agricultural College.
L. JEROME MANDEVILLE, Asso. in Arts (Holyoke J.C.) Engraver, Gravure Engraving Corp.
LAWRENCE D. MARCOTTE, M.S. (Ft. Hays, Kansas S.C.) Instr., Victoria Rural High School, Kansas.
FRANK J. MASSEY, JR., Ph.D. (California, Berkeley) Asso. Professor, University of California.
ROBERT MCFADDEN, B.A. (Queen's, Belfast) Teacher, Grosvenor High School, Belfast, Ireland.
J. PATRICK MCGAY, Student, University of Tulsa.
JAMES G. MCGREW, M.A. (Colorado S.C.) Teacher, Denver Public High Schools, Colorado.
BRENDA C. MCKEON, B.S. (Marymount) Grad. Asst., Georgetown University.
THOMAS G. MCLAUGHLIN, Student, University of California.
LOUIS D. MELNICK, Student, North Central College.
JOHN W. MEUX, M.A. (Arkansas) Grad. Asst., University of Florida.
ARNOLD L. MILNER, M.A. (Alabama) Instr., University of Alabama.
LOIS E. MINNING, Student, Oberlin College.
VESPER D. MOORE, Ed.D. (Michigan) Professor, Indiana State Teachers College.
FRED L. MORRISON, Student, University of Kansas.
JOHN C. MORRISON, Student, University of Santa Clara.
EDITH MOSS, M.A. (Maryland) Instr., Vassar College.
MICHAEL R. MULLEN, M.A. (Fordham) Instr., Fordham College.
JOSEPH P. NATHANSON, M.Ed. (Boston) Teacher, Weeks Junior High School, Massachusetts.
WARREN E. NUTE, Student, Stanford University.
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FRANCIS J. O'MEARA, B.S. (Creighton) Jr. Mathematician, Westinghouse Research Labs.
STANLEY S. PAGE, Student, University of Oregon.
THEODORE W. PALMER, M.A. (Johns Hopkins) Grad. Student, Harvard University.
JAMES T. PARENT, A.B. (Ricker) Grad. Student, University of Maine.
NANCY L. PARR, A.B. (Hanover) Teaching Fellow, University of Kentucky.
EMANUEL PARZEN, Ph.D. (California) Asso. Professor, Stanford University.
RICHARD H. PATTERSON, Student, University of British Columbia.
DIANE M. PAWLAK, Student, University of Wisconsin.
C. FREDERICK PETERS, A.B. (Vanderbilt) Mathematician, Naval Proving Ground.
ROBERT L. PEXTON, B.A. (California) Mathematician, Lawrence Radiation Lab.
ISAAC R. PFEIFFER, Student, Brooklyn College.

MARY L. POTTS, Student, University of Detroit.
WILLIAM D. POWERS, M.A. (Cornell) Teaching Asst., University of Tennessee.
JOHN B. PRATER, M.A. (Missouri) Instr., Southwest Missouri State College.
EUGENE V. PRICE, Student, North Dakota Agricultural College.
RICHARD H. PROSL, Student, College of William & Mary.
CALVIN M. PULLIAS, B.A. (Geo. Pepperdine) Grad. Student, University of Southern California.
SEYMOUR W. PUSTILNIK, M.A. (Bowling Green S.U.) Instr., New York City Community College of Applied Arts & Sciences.
DANIEL G. QUILLEN, Student, Harvard University.
DENNIS RADER, Student, Cooper Union.
PETER L. RENZ, Student, Reed College.
CARL E. RIECK, JR., B.A. (Yale) Grad. Student, Tufts University.
DOCK S. RIM, Ph.D. (Indiana) Research Asso., Columbia University.
MELVIN B. RINGEL, B.S. (M.I.T.) Jr. Instr., Johns Hopkins University.
CHARLES E. ROBE, Electronics Instr., Naval School of Interior Communications.
CHARLES D. ROBINSON, B.A. (Hardin-Simmons) Instr., Hardin-Simmons University.
R. LEWIS ROFMAN, M.S. in Ed. (Temple) Teacher, Olney High School, Philadelphia.
EDWARD ROSENBERG, M.A. (Illinois) Instr., East Los Angeles Junior College.
GEORGE M. ROSENSTEIN, JR., Student, Oberlin College.
CAPT. ROBERT C. ROUNDING, M.S. (Oklahoma S.U.) Instr., U. S. Air Force Academy.
MRS. MELBA L. ROY, M.S. (Howard) Mathematician, Army Map Service.
MRS. LAUREL R. RUCH, Student, Colorado College.
CHARLES T. RYAN, JR., B.S. (Canisius) Mathematician, Cornell Aeronautical Lab.
ABDEL S. SAID, M.S. (M.I.T.) Research Chemist, Columbia University.
SOCRATES W. SAUNDERS, Ph.D. (Pittsburgh) Professor, Morgan State College.
JAMES W. SAUVE, S.J., B.S. (Spring Hill) Grad. Student, Johns Hopkins University.
BERNADETTE SAVIGNAC, Student, Anna Maria College.
NATHAN S. SCARRITT, JR., M.A. (Oklahoma) Special Instr., University of Oklahoma.
WILLIAM H. SCHMIDT, B.A. (Columbia) Vice President & Chief Actuary, Mutual Life Ins. Co. of New York.
ALVIN L. SCHREIBER, B.S. (Johns Hopkins) Research Asso., Human Sciences Research, Inc.
CAROLE E. SCOTT, Student, Butler University.
JACK SEGAL, M.S. (Miami) Grad. Asst., University of Georgia.
STEPHEN B. SEIDE, Student, James Madison High School, Brooklyn, New York.
GEORGE SENGE, Student, University of California.
GEORGE C. SETHARES, B.Mus. (Boston) Grad. Student, University of Massachusetts.
DANIEL M. SHEEHAN, Student, College of William & Mary.
JAMES L. SIEBER, B.S. (Shippensburg S.T.C.) Grad. Asst., Pennsylvania State University.
NICHOLAS T. SIMOPOULOS, M.S. (Ohio S.U.) Vice President, Dayton Electronic Products Co.
DALE D. SIMS, Teacher, Nevada Union High School District, California.
SISTER M. DENIS, O.S.F., M.A. (New York) Chairman of Dept., St. Joseph's Girls High School, West New York, New Jersey.
SISTER MARY DE PAZZI, O.S.F., Ph.D. (Notre Dame) Head of Dept., Briar Cliff College.
SISTER MARY KENNETH, R.S.M., M.A. (Loyola) Head of Dept., Mother McAuley Liberal Arts High School, Chicago, Illinois.
SISTER MARY LAWRENCE, M.A. (Canisius) Teacher, Mt. St. Joseph Teachers College.
SISTER ROSE MARIAN, O.S.F., Instr., Ladycliff College.
MICHAEL S. SKAFF, B.S. (Michigan) Grad. Asst., University of Illinois.
WILLIAM E. SLESNICK, M.A. (Oxford) Instr., Dartmouth College.
GEORGE R. SMITH, A.B. (Princeton) Head of Dept., St. Paul's School, Concord, New Hampshire.
MARTIN R. SMITH, Stationary Engineer, International Breweries, Inc.
NATHANIEL B. SMITH, B.S.E. (Princeton) Teacher, Taft School, Watertown, Connecticut.
IAN N. SNEDDON, D.Sc. (Glasgow) Simson Professor, University, Glasgow, Scotland.
VIRGINIA A. SNOW, B.A. (Houghton) Teaching Fellow, University of Buffalo.
MAX A. SOBEL, Ph.D. (Columbia) Asso. Professor, Montclair State College.
DAVID C. SOMMER, Student, University of Minnesota.
JESSE C. SOWELL, B.S. (Carson-Newman) Instr., Elon College.
MRS. KATYE O. SOWELL, M.S. (South Carolina) Asst. Professor, Elon College.
WARREN G. SPARKS, A.M. (Boston) Instr., St. Michael's College, Vermont.
HARLAN C. STAMM, M.S. (Northern Illinois) Instr., Cerritos Junior College.
JERRY D. ST. CLAIR, B.S. (Central Michigan) Grad. Student, Michigan State University.
EDWIN I. STEIN, B.S. (Pennsylvania) Author, Philadelphia.
VINCENT STELMACH, B.A. (Connecticut) Mathematician, Daystrom Electric.
JOHN W. SUMMERS, M.A. (San Jose S.C.) Grad. Student, University of California.
COL. DANIEL N. SUNDT, M.A. (Pennsylvania) Professor, University of Delaware.
BARRY H. TALSKY, Student, Syracuse University.
MRS. OLGA TAUSSKY, Ph.D. (Vienna) Research Asso., California Institute of Technology.
WILLIAM C. TEACHOUT, JR., Student, Memphis State University.
EDWARD O. THORP, Ph.D. (U.C.L.A.) Moore Instr., Massachusetts Institute of Technology.
GEORGE C. TIBBETTS, S.B. (Harvard) Director of Research, Tibbetts Industries, Inc.
JOHN L. TILLEY, M.Ed. (Florida) Grad. Asst., University of Florida.
LEON TISDALE, M.S. (Oklahoma S.U.) Asst. Professor, Southern Methodist University.
BURNETT R. TOSKEY, Ph.D. (Washington) Instr., Seattle University.
CATHERINE B. TURIANO, B.A. (Good Counsel) Programmer, Sperry Gyroscope Co.
CAROL S. UMBREIT, B.S. (Pennsylvania S.U.) Grad. Student, University of Michigan.
KEENE C. VAN ORDEN, M.A. (Minnesota) Chairman of Dept., Central High School, San Angelo, Texas.
MELVIN N. VESELY, M.Litt. (Pittsburgh) Senior Math. Teacher, Carrick High School, Pittsburgh, Pennsylvania.
MARVIN E. WALDEN, Student, Wayne State University.
ROY F. WALLER, B.S. (East Central S.C.) Engineer Asso., Western Electric Co.
CDR. WILLIAM R. WALLIS, M.A. (Texas) United States Navy, Retired, Austin, Texas.
LETTY MAY WALSH, B.A. (Augustana, Illinois) Instr., University of Hawaii.
EVERETT L. WALTER, M.S. (New Mexico S.U.) Instr., New Mexico State University.
JOHN W. WARNER, M.S. (S.U. of Iowa) Instr., College of Wooster.
ROBERT H. WASSERMAN, Ph.D. (Michigan) Asst. Professor, Michigan State University.
BARTON WASSERMANN, A.B. (Temple) Engineer, Philco Corp.
GARTH H. WEBBER, A.B. (Indiana Central) Programmer, System Development Corp.
RONALD C. WEGER, Student, William Jewell College.
ALEXANDER WEINER, M.A. (Columbia) Instr., Hofstra College.
MRS. CAROLE H. WEISS, Student, Vassar College.
MARY W. WELTY, Student, Ohio Wesleyan University.
MRS. HELEN C. WHEELER, M.Ed. (Trinity) Asst. Professor, San Antonio College.
PATRICK J. WHELAN, Student, St. Francis Xavier University, Nova Scotia.
JOHN T. WHITE, M.A. (Texas) Lecturer, University of Texas.
ROBERT J. WHITLEY, A.A. (San Diego J.C.) Student, San Diego State College.
FRED H. WHITLOCK, Student, Agricultural and Technical College of North Carolina.
H. HOLLIS WICKMAN, Student, University of Omaha.
ERIC L. WILSON, B.S. (Westminister, Pennsylvania) Vanderbilt University.
CHARLES T. WOLF, B.S. (Millersville S.T.C.) Asst. Professor, State Teachers College, Shippensburg, Pennsylvania.

C. RAYMOND WOODROW, B.A. (Greenville) Grad. Asst., Oklahoma State University.
LAWRENCE E. WRIGHT, Student, Massachusetts Institute of Technology.
WALTER ZAYACHOWSKI, M.A. (Saskatchewan) Grad. Student, University of Alberta.
MARVIN ZELEN, Ph.D. (American) Mathematician, National Bureau of Standards.

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1959 by a mail vote of the membership of the Association in the Sections indicated:

Illinois	Rothwell Stephens, Knox College
Iowa	H. T. Muhly, State University of Iowa
Louisiana-Mississippi	Arthur Ollivier, Mississippi State College
Maryland-D. C.-Virginia	R. C. Yates, College of William and Mary
Michigan	R. M. Thrall, University of Michigan
Minnesota	J. M. H. Olmsted, University of Minnesota
Philadelphia	Albert Wilansky, Lehigh University
Southern California	P. B. Johnson, Occidental College
Texas	W. T. Guy, Jr., University of Texas

In two sections, votes were received from more than 50% of the membership. These sections were Iowa with 58% and Louisiana-Mississippi with 52%.

L. J. MONTZINGO, *Associate Secretary*

THE NEW SECRETARY OF THE ASSOCIATION

Professor Henry L. Alder of the University of California, Davis, has been elected by the Board of Governors to the office of Secretary of the Association for the five-year term 1960–1964. He will begin to assume some of the responsibilities of his office after September 1, 1959, and it is expected that the transfer of all secretarial duties will be completed by January 1, 1960.

Professor Alder has been a member of the Association since 1950. He has served as a member of the Committee on High School Contests. In the Northern California Section of the MAA he has served as Vice-Chairman and Chairman as well as the first Chairman of the Section's Committee on Contests and Awards. He has been active in organizing the Program of Visiting Lecturers to Secondary Schools in Northern California and has himself served as a Visiting Lecturer. He has been President of Mu Alpha Theta since its organization in 1957; his term as President expires on July 31, 1959. He is Vice-Chairman of the Board of Governors of the Pacific Journal of Mathematics for a three-year term beginning June 1, 1957.

The By-Laws of the MAA were amended last January at the Philadelphia meeting to provide for the separation of the offices of Secretary and Treasurer. Since then I have been serving as Acting Secretary as well as Treasurer. It is with great pleasure that I am turning over some of my duties to Dr. Alder. A statement will appear in one of the early issues of the MONTHLY regarding the division of responsibilities between the Secretary and the Treasurer, in so far as this concerns the membership of the Association.

HARRY M. GEHMAN
Acting Secretary

THE TWENTIETH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The twentieth annual William Lowell Putnam Mathematical Competition will be held on Saturday, November 21, 1959. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of

America and is open to regularly enrolled undergraduate students in universities and colleges of the United States and Canada who have not yet received a college degree.

Application blanks will be mailed about October first to the regular mailing list. If an application blank is not received by October 15, you may secure one by writing the director, Professor L. E. Bush, 301 Merrill Hall, Kent State University, Kent, Ohio. Your application must be filed with the director not later than November 1, 1959. For further details of the examination and the list of prizes (including the $3000 scholarship at Harvard), see the announcement which will be mailed out along with the application blank.

Reports of the previous competitions and the examinations will be found in this MONTHLY for May 1938, 1939, 1940, 1941, 1942; October 1946; August-September 1947; December 1948; August-September 1949, 1950, 1951; October 1952, 1953, 1954, 1955; December 1956; August-September (announcement of winners) and November (questions and solutions) 1957; August-September, 1958; and this issue.

CARUS MONOGRAPH NUMBER 12

A new Carus Monograph is now in press. Publication is expected in November. Its title is:

MONOGRAPH 12: *Statistical Independence in Probability, Analysis and Number Theory*, by Mark Kac.

Each member of the Association may purchase one copy of this Carus Monograph at the special price of $1.75. Orders accompanied by payment should be addressed to: Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

Additional copies of Monograph 12 for members and copies for nonmembers may be purchased at $3.00 from John Wiley and Sons, 440 Fourth Avenue, New York 16, New York.

Professor Kac's monograph is an expanded version of the three Hedrick Lectures he delivered at the 1955 Summer Meeting of the Mathematical Association of America. The monograph is designed to illustrate how simple observations can be made the starting point of rich and fruitful theories and how the same theme recurs in seemingly unrelated disciplines.

An elementary but thorough discussion of the game of "heads or tails," including the normal law and the laws of large numbers, is presented in a setting in which a variety of purely analytic results appear natural and inevitable. The chapter "Primes play a game of chance" uses the same setting in dealing with problems of the distribution of values of arithmetic functions. The final chapter "From kinetic theory to continued fractions" deals with a spectacular application of the ergodic theorem to continued fractions. A novel feature is the inclusion of a large number of problems.

The author is professor of mathematics at Cornell University and is the winner of the 1950 Chauvenet Prize of the Association.

THE FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The thirty-sixth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at the Buena Vista Hotel, Biloxi, Mississippi, on February 13–14, 1959, with Delta State College acting as host institution. The Friday afternoon meeting was held in two concurrent sessions. The Rev. T. F. Mulcrone, Louisiana Vice-Chairman, and Professor A. A. Ritchie presided. Professor C. G. Killen, Chairman of the Section, presided at the Friday evening and Saturday morning sessions. There were 153 persons registered, including 61 members of the Association.

The following officers were elected for the coming year: Chairman, Professor S. B. Murray, Mississippi State University; Vice-Chairman for Louisiana, Professor T. K.

Maddox, Southeastern Louisiana College; Vice-Chairman for Mississippi, Professor W. M. Sanders, Mississippi Southern College; Secretary-Treasurer, Professor T. L. Reynolds, Millsaps College.

At the business meeting a report on the progress of the high school mathematics contests was given by Professor Noel Childress, Chairman of the Committee on Contests. Professor Z. L. Loflin, Sectional Governor, reported on the recent meeting of the Sectional Officers.

The invited speaker for the meeting was Professor W. V. Parker of Alabama Polytechnic Institute. His lecture on Friday evening was entitled *Matrices and Polynomials* and his Saturday morning address was on *Commutative Matrices*.

The following papers were presented:

1. *The LSU year-long Mathematics Institute sponsored by NSF*, by Professor H. T. Karnes, Louisiana State University.

Beginning with the fall semester of 1959, Louisiana State University will have a year-long Mathematics Institute in mathematics. The program is sponsored by the National Science Foundation. This program will include the academic year, plus the summer semester of 1960. With this arrangement, the participants will be able to complete a program leading to the Master's degree. Forty participants will be in the program.

A specially designed curriculum of twenty-one semester hours has been prepared for the participants. This curriculum is designed to give the participants a broad background for the field of mathematics, and to consider topics which are likely to appear in the secondary school mathematics program in the near future. Those desiring a Master's degree will have to complete an additional nine semester hours of study in mathematics or related fields.

2. *Cyclo subpedal curves*, by Professor V. B. Temple, Millsaps College.

If $H(X_1, Y_1)$ and $E(X_2, Y_2)$ are two points on the hypocycloid and epicycloid respectively, corresponding to any given angle θ, then $P_1(X_1, Y_2)$ and $P_2(X_2, Y_1)$ are points on two cyclo subpedal curves. Since H and E are pedal curves, P_1 and P_2 are subpedals. In like manner the subpedals of the hypo- and the epirose curves are defined. (See this MONTHLY, vol. 59, p. 67.) All of this family of eight curves are governed by a single parametric angle θ; that is, if θ changes, all of the eight points move in harmony, each tracing its own distinct curve. A chart showing the curves P_1 and P_2 of the four cusped hypo- and epicycloids was presented.

3. *An experiment in readiness for logical thinking and demonstrative geometry*, by Professor G. J. Corley, Northwestern State College of Louisiana.

A discussion of an experiment in teaching some lessons in methods of reaching conclusions; intuition, inductive reasoning, and deductive reasoning; and an introductory unit in demonstrative geometry to sections of students from the sixth grade to the tenth grade.

4. *A set of postulates for trigonometry*, by Professor Arthur Ollivier, Mississippi State University.

The object of this paper is to exhibit a set of postulates from which all the properties of analytical trigonometry may be derived. The sine and cosine functions are defined in terms of a real variable without introducing the concept of an angle. Numerical aspects of the subject form an example of the theory.

5. *Remarks on inverse trigonometric functions and integration by trigonometric substitution*, by Professor W. E. Koss, Louisiana Polytechnic Institute.

The procedure in integration by trigonometric substitution as described in many calculus texts is fallacious, or at the least highly misleading. To rectify this it is essential to introduce into trigonometry meanings for, say, $\sin x = a$ and $\sin^{-1} a$, where $|a| \leq 1$, which are analogous to meanings

assigned in algebra to say, $x^n=a$ and $\sqrt[n]{a}$, where $a>0$. To accomplish this it is proposed that symbols for the inverse trigonometric functions be defined to insure first, single-valuedness and secondly, if possible, continuity.

Furthermore $\cos x=\sqrt{1-\sin^2 x}$ is not an identity unless the domain is restricted properly. In most calculus texts it is used as an identity without any mention of any restrictions. With the inverse functions properly defined and inverse trigonometric substitution used these errors may be avoided.

6. *On a certain property of infinite series*, by Professor R. D. Boswell, Jr., Mississippi State University.

The question answered in this paper arose in the teaching of infinite series in the calculus course. Consider the series $\sum_{k=1}^{\infty}u_k$. It is pointed out that a necessary and sufficient condition that a series converge is that (a) $\lim_{k\to\infty}u_k=0$ and (b) there exists a sequence of positive integers $n_1, \cdots, n_i, \cdots$ such that $0<n_i-n_{i-1}\leq M$ where M is a positive real number and such that the subsequence $S_{n_1}, \cdots, S_{n_i}, \cdots$ of the sequence of partial sums converges. An example is exhibited to show that the bound M on the differences n_i-n_{i-1} is necessary.

7. *On algebra, trigonometry, and analytic geometry*, by Professor B. E. Mitchell, Louisiana State University.

Mathematics is supposed to be precise. Elementary textbooks in algebra, trigonometry, and analytic geometry are usually not precise, at times. This statement is clarified by illustrations from polynomials, factors, nth roots, equations, identities, and polar coordinates.

8. *G-spaces with the Barilian metric*, by Professor W. M. Sanders, Mississippi Southern College.

A Barilian space is defined to be the interior of a simple, closed curve J in E^2 remetrized with the Barilian metric $\delta(a, b)=\log(\max_{p\in J}ap/bp)+\log(\max_{q\in J}bq/aq)$. If the defining curve J is convex, then the Barilian space is a straight G-space.

9. *A characterization of the symmetric group*, by Professor J. D. Gilbert, Louisiana Polytechnic Institute.

After a preliminary theorem, the following result is presented:

Let G be a group transitive on $M=(a_1, \cdots, a_n)$, and let H_i be the subgroup of G which leaves $a_1, \cdots, a_i$ fixed for $i=1, \cdots, n$. If $H_i\neq H_j$ for all pairs i, j such that $i\neq j$ and $i<n-1$, then G is the symmetric group on n elements. Conversely, if G is the symmetric group on n elements, then $H_i\neq H_j$ for all pairs $i\neq j$ and $i<n-1$.

10. *A note on certain approximations and their corresponding curves*, by Professor T. A. Bickerstaff, University of Mississippi.

Certain approximations were developed for distances, and their curves were described. Using the arithmetic mean of a geometric predecessor and successor of value to be approximated, the following result was obtained:

$$\sqrt{x^2+y^2}\sim(8x^4+8x^2y^2+y^4)/4x(2x^2+y^2), \qquad x\geqq y.$$

Another adaptation led to the following interesting formula, with x^2 equal to the largest square in N, $D=|N-x^2|:\sqrt{N}\sim(8x^2N+D^2)/4x(x^2+N)$. Curiously this represents a good approximation whether x^2 is greater than N or less than N so long as it is relatively near.

11. *An integration formula for the hyperbolic partial differential equation with region of integration exterior to the characteristic cone*, by Professor Paul Sanders, Louisiana Polytechnic Institute.

At the annual meeting of the American Mathematical Society in Pittsburgh in December

1954, R. B. Deal and E. W. Titt presented a method of deriving an integration formula for the hyperbolic equation with region of integration interior to the characteristic cone. Their method is applied to the problem of deriving an integration formula for the hyperbolic equation with region exterior to the characteristic cone.

12. *A characterization of compactness for regular spaces*, by Professor R. W. Bagley, Mississippi Southern College.

Let β be a family of finite character contained in 2^{α} where α is the collection of open sets in a topological space. We consider the following conditions, where $M(\beta)$ denotes the set of maximal elements of β. (1) If $\Omega \in M(\beta)$ then $U \notin \Omega$ and $V \notin \Omega$ implies that $U \cap V \notin \Omega$. (2) If $\Omega \in M(\beta)$ then $U \notin \Omega$ and $U \subset V$ implies that $V \notin \Omega$. (3) If $\Omega \in M(\beta)$ then $U \in \Omega$ and $V \in \Omega$ implies that $U \cup V \in \Omega$. (4) $\bigcap_{\Omega \in M(\beta)} \Omega = \{\phi\}$, the set whose only element is the null set.

THEOREM. *If X is a regular T_1 space with open subbase α_0 and there is a family β of finite character satisfying conditions* 1) *through* 4), *then X is compact if and only if no covering of X by sets of α_0 is a member of β.*

LEMMA. *If X is a topological space with open subbase α_0 and there is a family β of finite character satisfying conditions* 1) *and* 2), *then no open covering is a member of β if and only if no covering by sets of α_0 is a member of β.*

By taking β to be the family of open collections which have no finite subcollection covering X we have the following as a corollary to the Lemma.

(Alexander) A topological space X with open subbase α_0 is compact if and only if every covering by sets of α_0 has a finite subcovering.

S. R. KNOX, *Acting Secretary*

THE MARCH MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held on March 28, 1959 at Michigan State University, East Lansing, Michigan in conjunction with the meeting of the Michigan Academy of Science, Arts, and Letters. Professor G. Y. Rainich of the University of Michigan and Notre Dame University presided at both the morning and afternoon meetings and at the luncheon-business meeting. A total of 60 persons attended the meeting, including 44 members of the Association.

A nominating committee consisting of Professors C. H. Butler, Chairman, and W. K. Folley proposed Professor W. D. Baten of Michigan State University for Chairman, Professor E. E. Moise of the University of Michigan as Vice-Chairman and Professor L. E. Mehlenbacher of the University of Detroit as Secretary-Treasurer. The slate was elected unanimously.

Professor J. S. Frame of Michigan State University gave the Presidential address on *A bridge to relativity theory* before the Michigan Academy of Science, Arts, and Letters.

The committee on the Mathematics Prize Competition consisting of Professors F. L. Celauro, Chairman, Central Michigan University; A. J. Lohwater, University of Michigan; A. W. Jacobson, Wayne State University; W. E. Deskins, Michigan State University; and J. B. Eckstein, University of Detroit, gave a report of the second annual prize competition held in over 400 Michigan High Schools to over 8000 participating students.

The following papers were presented:

1. *Generalized derivatives*, by Professor Jacob Korevaar, University of Wisconsin. (By invitation).

Two wild animals in the mathematical zoo are the stickle-back (a continuous nowhere differentiable function), and the perfect sneak (a continuous strictly increasing function with zero derivative almost everywhere). In the physicist's jungle the δ animal, although untamed there, performs useful services. By now δ and the related improper animals have been completely do-

mesticated in mathematics. They have been caught in the theory of distributions. This theory is explained and its applications to these functions demonstrated.

2. *Comparison of estimates of the total number of cows in a certain area of Michigan by the ratio and the non-ratio methods*, by Professor W. D. Baten, Michigan State University.

The ratio method is as follows. Let x_i and y_i respectively represent the number of acres and cows in the ith farm. Let the ratio, R_n, be defined as $R_n = \sum y / \sum x$ where the numerator is the sum of the cows in a random sample and the denominator is the number of acres in the farms in the sample.

An estimate of the total number of cows in the area is $y_R = R_n X$ where X is the total number of acres in the area. This estimate is compared with another estimate that does not involve the relationship between x and y. The standard deviations of these estimates are used in the comparison.

3. *Three-dimensional poker hand frequencies by combinatorial analysis and by I.B.M. 704*, by Professor W. W. Funkenbusch, Michigan College of Mining and Technology.

The three dimensional deck consists of 64 cards, uniquely determined by 4 suits, 4 denominations, and 4 colors. In three dimensional poker, two cards are of the *same kind* if they have two common coordinates. Any one particular card can be used to help form only one unit. The numbers of Four of a Kind, Full House, Three of a Kind, Two Pair, One Pair, and No Pair hands can be determined analytically by decomposition into 21 *linkage types*. An I.B.M. 704 program for frequency determination, due to D. L. Shell of General Electric, is given.

4. *The development of the mathematical theory of linear perspective*, by Professor P. S. Jones, University of Michigan.

The high-lights of the development were presented from its beginning with Leone Battista Alberti in 1435 through the use of double orthogonal projection by Piero della Francesca and Albrecht Durer to the generalizations and extensions of Brook Taylor and J. H. Lambert. Connections with coordinates, descriptive and affine, as well as projective geometry were pointed out as well as suggestions for possible uses of these materials in teaching—especially in teacher preparation.

5. *A numerical methods program at General Motors Institute*, by Mr. M. L. DeMoss, General Motors Institute.

This paper outlines four intensive training courses offered at General Motors Institute for corporation people actively engaged in the solution of engineering problems using digital computers. The courses include iterative solution of algebraic and transcendental equations, matrices, interpolation and curve fitting, and numerical solution of differential equations. The author indicates that three of the subject courses are to be integrated and offered as an elective to General Motors Institute engineering students. He suggests that such a course could profitably be offered to mathematics majors by any liberal arts college not having a digital computer.

6. *Some applications of Boolean algebra to the nervous system*, by Sister Marian Joan, S.N.J.M., University of Detroit, introduced by the Secretary.

In this paper it was shown that Boolean algebra can be applied to nerve networks in order to analyze patterns of action known to occur, as in the phenomenon described by McCulloch and Pitts (*Bulletin of Mathematical Biophysics*, V. 1943) by which a person feels heat when he touches a cold object momentarily, but feels cold when he holds the object. It was also shown how Boolean algebra can be applied to testing neural models for fulfillment of the biological requirements for such models, by testing Piéron's neurological explanation of a possible mechanism for color vision published in 1952.

7. *Using $y = mx + b$ to score mathematics tests*, by Professor H. B. Anderson, Michigan College of Mining and Technology.

Linear grading is often discarded because the application is cumbersome or the results are unreliable. This paper suggests that the slope, determined by a minimum and a maximum performance, be set on an ordinary slide rule to simplify grade computations. A method of quiz construction based on core material is suggested for use in conjunction with linear grading. Using a departmental final involving about 400 students as a standard of comparison, linear grading gave a correlation coefficient of .85 against .73 for conventional methods as applied to mathematics students at the Michigan College of Mining and Technology.

8. *Capacity in two and three dimensions*, by Professor J. L. Ullman, University of Michigan.

Volume is a function on sets in three space, and area is a function on sets in the plane. They are related by the simple observation that the volume of a right cylinder of unit height is equal to the area of the cross section.

Capacity is a function on sets in three space, and there is another notion, also called capacity, defined for sets in the plane. The simple expedient that relates area and volume fails for the two notions of capacity, and a more elaborate procedure is developed that achieves this end. This procedure enables one to use intuition of electrostatics, which is closely related to the capacity of three dimensional sets, to the capacity of sets in the plane.

F. A. BEELER, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The thirty-ninth regular meeting of the Southern California Section of the Mathematical Association of America was held at the University of Redlands, Redlands, California, on March 14, 1959. Professor P. J. Kelly, Chairman of the Section, presided. The registered attendance was 132, including 96 members of the Association.

At the business meeting Professor R. A. Dean, Chairman of the Nominating Committee, reported that the following officers were elected by mail ballot for the next academic year: Chairman, Professor D. H. Hyers, University of Southern California; Vice-Chairman, Professor R. C. James, Harvey Mudd College; Secretary-Treasurer, Mr. R. B. Herrera, Los Angeles City College. The Nominating Committee appointed the following members to the Program Committee for the coming year: Professor Tom Apostol (Chairman), California Institute of Technology; Professor C. V. Holmes, San Diego State College; Professor P. J. Kelly, University of California, Santa Barbara; Mr. Fred Marer, Los Angeles City College; Dr. R. W. Rector, Ramo-Wooldridge Corporation. In regard to contests, the following motion was adopted at the business meeting: "The national office is hereby requested to conduct the national contest in the Southern California Section, starting in 1960, in those cities and rural areas in which local contests are not operating. A list of such areas will be furnished to the national office by the Section." (There are at least five major contests in areas of the Section at present.)

The following program was presented:

1. *Man versus machines*, by Dr. E. M. McCormick, Naval Ordnance Laboratory, Corona, California, introduced by the Secretary.

The relationships between mathematicians and automatic digital computers were considered. The emphasis was on the fact that these computers were developed originally as strictly mathematical tools, but they are now being used by a number of other disciplines. This has resulted in applications to translation of languages, retrieval of information, abstracting of articles, and other interesting fields. Anthropomorphic designations for computers and some of their functions were also considered.

2. *The geometry of numbers*, by Professor Kenneth Rogers, University of California, Los Angeles, introduced by the Secretary.

Minkowski's theorem on linear forms asserts that if we have n linear forms of determinant

one, then integer values not all zero can be found for the variables so as to make all the forms less than or equal to *one* in absolute value. It was proved by Hajos, that if the forms cannot all be made less than *one* then they are equivalent to a set whose matrix has *ones* down the diagonal and *zeros* above it. A new method for trying to prove Hajos' theorem was indicated.

3. *Decomposition*—one-hour invited address—by Professor Hans Zassenhaus, McGill University and California Institute of Technology.

The forms under which the Clavius postulate, that the whole is equal to the sum of its parts, appear in various mathematical fields, were traced.

4. *Auditory images of non-accelerated sources of sound*, by Professor Hugh Hamilton, Pomona College.

When an airplane traveling in a straight line at constant speed occupies any position P it reports itself to the ears of an observer on the ground at O (if at all) as being at some earlier position I (or two such positions, I_1 and I_2). Angular relationships at O between P and I (or I_1 and I_2) are developed, among them this: at supersonic speeds (when there is for a while no "I" but ultimately both "I_1" and "I_2") the average of the angular positions of I_1 and I_2 at O is always 90° behind the angular position of P at O. The techniques used are exclusively those of tenth grade geometry.

5. *Professor Begle's School Mathematics Study Group*, by Mr. L. C. Lay, Pasadena City College, and Mr. William Wooton, Pierce Jr. College, Los Angeles.

The scope of the School Mathematics Study Group and its objectives for the improvement of the teaching of mathematics in the secondary schools were presented. The progress of the Monograph Project of this group was reported, and mention was made of the plans for improving the training of mathematics teachers. The experimental textbooks, which they are writing for grades seven through twelve, were discussed.

6. *A stability theorem for non-linear ordinary differential equations*, by Professor H. A. Antosiewicz, University of Southern California.

The title refers to a theorem, essentially due to Malkin, on the uniform-total stability of the solution $y(t)\equiv 0$ of a vector differential equation (1) $y'=f(t, y)$ with $f(t, 0)\equiv 0$. It states that if $f(t, y)$ is Lipschitzian in y then uniform-asymptotic stability of $y(t)\equiv 0$ implies uniform-total stability; and if $f(t, y)$ is linear in y then exponential-asymptotic stability of $y(t)\equiv 0$ is equivalent to uniform-total stability. The importance of this theorem was discussed for problems in which the asymptotic behavior of the solutions of an equation (2) $x'=f(t, x)+g(t, x)$ with $g(t, x)$ small, in some sense, for large t, was studied by comparing the solutions of (2) with those of (1).

R. B. HERRERA, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The 46th meeting of the Iowa Section of the Mathematical Association of America was held at Iowa Wesleyan College, Mt. Pleasant, Iowa, April 17, 1959. Professor E. N. Oberg, Chairman of the Section, presided. Total attendance was 50, including 23 members of the Association.

Routine business was considered during the afternoon meeting. The following officers were elected: Chairman, Professor J. J. L. Hinrichsen, Iowa State College; Vice-Chairman, Professor H. C. Trimble, Iowa State Teachers College; Secretary-Treasurer, Professor E. L. Canfield, Drake University.

The following papers completed the program:

1. *Wedges: A discussion of partially ordered linear spaces*, by Professor M. M. Day, University of Illinois. (By invitation).

The useful properties of a partial order relation ($\geq$) among elements of a linear space L can

be characterized by geometric properties of the set P of elements ≥ 0. Examples were given. Fixed vectors and fixed directions of linear operators preserving order were discussed next. Monotone linear functionals, their monotone linear extensions, and the geometric interpretation of the theorems which are known were mentioned.

2. *A necessary condition for the completeness of a family of probability measures*, by Professor R. V. Hogg, State University of Iowa.

A family $\{F(x;\theta);\theta\in\Omega\}$ of distribution functions is complete if $\int_{-\infty}^{\infty}u(x)dF(x;\theta)=0$, for all $\theta\in\Omega$ and any real $u(x)$ that is absolutely integrable with respect to $\{F\}$, implies that $u(x)=0$ almost everywhere in $\{F\}$. Let $\phi(t;\theta)=\int_{-\infty}^{\infty}\exp(itx)dF(x;\theta)$ be the characteristic function of $F(x;\theta)$ A family $\{\phi(t;\theta);\theta\in\Omega\}$ is c-complete if $\int_{-\infty}^{\infty}[\alpha(t)+i\beta(t)]\,\phi(t;\theta)dt=0$ for all $\theta\in\Omega$ and any real $\alpha(t)$ and $\beta(t)$ that are absolutely integrable, implies that $\alpha(t)=0$ and $\beta(t)=0$ almost everywhere Lebesgue. A necessary condition that the family $\{F(x;\theta);\theta\in\Omega\}$ be complete is that the family $\{\phi(t;\theta);\theta\in\Omega\}$ be c-complete.

3. *Method of solving a class of mixed boundary value problems*, by Professor Don Kirkham, Iowa State College.

It is shown, by solving two practical potential problem examples, how mixed boundary value functions in a class of problems may be developed, over a boundary, into an infinite series of harmonic functions, leading to the solution. The development is accomplished by introduction of an auxiliary infinite series valid over the portion of the boundary where the normal derivative is given. One of the examples is a two-dimensional one for which a solution can also be obtained by conformal transformations. The solution by the two methods agree. The other example involves Bessel functions. The method of this paper does not require the breaking up of the region in question into auxiliary spaces.

4. *Waring's problem, modulo p, and the representation symbol*, by Sister M. Anne Cathleen Real, Marycrest College, presented by Miss Nancy Ketelaar, introduced by the Chairman.

The representation symbol $[a, b, c]$ is the statement that an integer of n-ic type a is congruent to the sum of an integer of n-ic type b and an integer of n-ic type c. The symbol is extended to include any definite number of elements. New properties, together with a list of symbols involving the n-ic types of specific integers, are derived for use in studying Waring's problem, modulo p, for a particular exponent n. Let $T_p(n)$ be the least number such that every integer is congruent to the sum of $T_p(n)$ or fewer n-ic residues. Then for primes of the form $22k+1$, $k>3$, $2\leq T_p(11)\leq 4$.

5. *An Iowa mathematics test*, by Professor O. C. Kreider, Iowa State College.

An Iowa college mathematics test was constructed for Iowa high school seniors, the results to be used by the Iowa private and state colleges for admissions, scholarships, and placement. Three members of the test construction committee were from private colleges and three from state colleges. Fifty per cent of the items test mechanics, the other fifty per cent require thinking. The test items were selected from arithmetics, algebra (I and II), geometries (plane, solid and analytic), trigonometry, and miscellaneous subjects (statistics, sets, and logic).

6. *An explicit example*, by Professor A. T. Craig, State University of Iowa.

An elementary explicit example of a nonnormal bivariate distribution that has normal marginal distributions is given. Let $f(x, y)=f_1(x, y)+kf_2(x, y)$ where $-\infty<x, y<\infty$, $0<|\rho|<1$, $f_1(x, y)=[2\pi(1-\rho^2)^{1/2}]^{-1}\exp[-(x^2+y^2-2\rho xy)/2(1-\rho^2)]$, and $f_2(x, y)=f_1(x, y)[x^2-(1+\rho^2)xy/\rho+y^2-(1-\rho^2)/2]\exp[-(x^2+y^2-2\rho xy)/2(1-\rho^2)]$. Now $|f_2(x, y)/f_1(x, y)|<M$ so that $f_1(x, y)[1+kf_2(x, y)/f_1(x, y)]\geq 0$ if $|k|<1/M$. Moreover, $\int_{-\infty}^{\infty}f_2(x, y)dx=\int_{-\infty}^{\infty}f_2(x, y)dy=0$ so that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x, y)dydx=1$. That is, $f(x,y)$ is a nonnormal joint probability density function but each marginal distribution is normal. If one wishes to take $\rho=0$ in $f_1(x, y)$, he may at the same time replace, in the definition of $f(x, y)$, $f_2(x, y)$ by $f_1(x, y)(xy)\exp[-(x^2+y^2)/2]$.

7. *Some integrals and series involving Legendre associated functions that arise in added mass theory*, by Professors L. Landweber and Matilde Macagno, Iowa Institute of Hydraulic Research, State University of Iowa, presented by Professor Macagno.

The added mass of a sphere performing high frequency horizontal oscillations when half submerged in a liquid has been evaluated. The problem consists of solving Laplace's equation with the velocity potential vanishing on the free surface. The value of the added mass coefficient, *i.e.*, the ratio of the kinetic energy of the fluid to the kinetic energy of the fluid mass displaced by the hemisphere, is obtained by solving finite and infinite series containing integrals of the Legendre associated functions. The coefficient is found to be $C_H = 4/\pi - 1 = 0.2732 \cdots$.

8. *Shears and inequalities*, by Professor Bernard Vinograde, Iowa State College.

The purposes of this note are (1) to establish the non-cycling of a shear-translation procedure for solving linear inequalities, and (2) to compare the logic of a test for inconsistency with that of the elimination method as used by H. W. Kuhn (*Solvability and consistency for linear equations and inequalities*, this MONTHLY, vol. 63, 1956, pp. 217–232).

9. *Unconditional probability*, by J. G. Baron, M.D., Iowa City, Iowa.

The concept of conditional probability is used to calculate an unknown probability from known ones. This is done by a certain restriction of the possible cases. The usual expressions are *if known* or *if given*. A simple example is presented in which such a formulation leads to a contradiction. This contradiction is discussed. It is shown that the expressions mentioned may involve a volitional act of an informer. This makes the use of the concept of mathematical probability impossible. If the expression *known as a result of a random experiment* is substituted, then the paradox is eliminated.

10. *Note on a limiting distribution*, by Professor D. A. Jones, State University of Iowa, (read by title only).

Let X be a real random variable of the continuous type possessing a probability density function (*pdf*), $f(x)$, which is continuous on the interval (a, b), positive almost everywhere [Lebesgue] on $[a, b]$ and zero outside of $[a, b]$. Let X_1 and X_n denote the smallest and largest items, respectively, of a random sample of size n from the distribution. Let $Q(x)$ be a measurable function essentially bounded [Lebesgue] on (a, b); we may assume $E[(Q(X)] = 0$ and $E[\overline{Q(X)}^2] = 1$. Then $y = n\int_a^{X_1} f(x)dx$, $v = n\int_{X_n}^b f(x)dx$, and $z = (1/\sqrt{n}) \sum_{\alpha=1}^n Q(X_\alpha)$ have a limiting distribution given by the joint *pdf*: $(1/\sqrt{(2\pi)})\exp(-y-v-z^2/2)$ for $y>0$, $v>0$, all z; and 0 elsewhere.

11. *An application of generalized means*, by Professor S. D. Nolte, Iowa State College, introduced by the Secretary.

The generalized mean $M(x, y)$ was defined to be $\psi^{-1}[p\psi(x)+q\psi(y)]$ where $p, q>0$, $p+q=1$ and $\psi(x)$ is monotone and continuous. This mean was applied to the second difference $\Delta^2(f; x, h) = f(x+h)+f(x-h)-f(x)$ to form a generalized second difference $\Delta^2_\psi(f: x, h) = M_\psi[f(x+h), f(x-h)] - f(x)$.

A study was made of functions whose generalized second differences satisfy certain conditions. Maxima of classes of generalized quasismooth functions were examined.

E. L. CANFIELD, *Secretary*

THE APRIL MEETING OF THE KANSAS SECTION

The forty-fourth annual meeting of the Kansas Section of the Mathematical Association of America was held at Marymount College, Salina, Kansas, on April 11, 1959, in conjunction with the annual meeting of the Kansas Association of Teachers of Mathematics. Professor P. S. Pretz, Chairman, presided at the sessions. Of the 159 persons registered, 78 are members of the Association.

The following officers were elected for one year terms: Chairman, Professor J. D. Haggard, Kansas State College of Pittsburg; Vice-Chairman, Professor W. D. Bemmels, Ottawa University; Secretary-Treasurer, Miss Helen Kriegsman, Kansas State College of Pittsburg.

At the joint session, held in the morning, Professor B. W. Jones, University of Colorado, delivered a paper entitled *A method for solving some quadratic Diophantine equations.*

The following short papers were presented at the afternoon session:

1. *A geometric number system*, by Sister Helen Sullivan, Mount St. Scholastica College.

A geometric number system is developed by employing a procedure parallel to that used in the construction of an algebraic number system. Equivalence classes of integers, rationals and reals are defined in terms of appropriate flat pencils. The operations of addition and multiplication are achieved by setting up flat pencils in an involution. Comments (without proofs) were given on the field properties in each system. Carefully drawn illustrations for adding and multiplying the lines representing equivalence classes of integers, rationals and reals were shown by use of an opaque projector. The isomorphisms present were also discussed.

2. *Series transformations*, by Professor J. R. Hanna, University of Wichita.

A brief review of the series transformation, $M\{F(t)\}=f(s)=\sum_{t=0}^{\infty} K(s, t)F(t)$ was outlined. Special attention was given to a few properties of the series transformation used by Laplace. In this transformation, the kernel $K(s, t)=s^t$ was used. By employing the Cauchy product, a convolution theorem was established.

3. *Curves in Euclidean space*, by Professor W. L. Stamey, Kansas State University.

Some possible definitions for the concept of a curve are given. The applicability of each possible definition is illustrated by examples of point sets which, according to the given definition, would be curves and examples of point sets which would not be curves. All the examples are subsets of Euclidean two- or three-dimensional space. The trial definitions lead to the definition of a curve as a one-dimensional continuum with dimension defined in the sense of Menger and Urysohn.

4. *A report on the current offerings in mathematics through the calculus at the various colleges in the state of Kansas*, by Professor C. B. Read, and Professor Agnes Nibarger, University of Wichita.

A questionnaire sent Kansas colleges, asking about mathematics courses through calculus indicates relatively rare offering of combination algebra-trigonometry-analytics but considerable offering of analytics-calculus. Schools with such offering form a minority, but *students involved* a majority. Large schools often offer the fused course in addition to separate courses. Some possible reasons for the trend toward combined courses are: a saving in class time spent developing subject matter, a saving in money spent for textbooks, a reduction in teaching load. A discussion was given of actual achievements made by the change and of some problems likely to occur.

5. *The modernization of mathematics*, by Professor O. J. Peterson, Kansas State Teachers College.

The paper presented the nature and purposes of modern secondary mathematics and the work of supporting groups, with special reference to the School Mathematics Study Group, of which Professor E. G. Begle, Yale University, is Executive Director. Questions discussed related to (1) the need for modernized elementary college texts, (2) the need for the establishment of institutes for capable college teachers with conventional mathematics backgrounds, and (3) the articulation of present college freshman courses with preparation in modern high school mathematics.

Attention was called to the urgency of solutions of these and other problems which colleges

will face when considerable numbers of high school graduates enroll with preparation in the new mathematics.

HELEN KRIEGSMAN, *Secretary*

THE APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The eighteenth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Polytechnic Institute of Brooklyn on April 18, 1959. Dr. Ernst Webber, President of Polytechnic Institute of Brooklyn gave the address of welcome and discussed *The impact of mathematics on engineering education.* Professor Azelle B. Waltcher, Collegiate Vice-Chairman of the Section, presided at the morning session and Dr. R. N. Walter, High School Vice-Chairman, presided at the afternoon session which was a panel discussion on *The role of geometry in secondary and higher education.* There were 144 persons in attendance, including 79 members of the Association.

Professor James Eastham, Chairman of the Section, presided at the business meeting. Reports were given by Professor Eastham on the Speaker's Bureau; by the Governor, Professor Jewell H. Bushey; the Secretary, Professor June Jensen; and Professor Charles Salkind, for the Committee on Contests and Awards. The following officers were elected: Chairman, Professor Azelle B. Waltcher, Hofstra College; Vice-Chairmen, Professor Jules Russell, Polytechnic Institute of Brooklyn, and Mr. George Grossman, William Howard Taft High School; Secretary, Professor Mary Dolciani, Hunter College; and Treasurer, Mr. Aaron Shapiro, Brooklyn College.

The following papers were presented at the meeting:

1. *Some topics in set theory*, by Professor Smbat Abian, Queens College.

Set theory is found to be essential in formulating the basic concepts of almost every mathematical discipline. The axiom system Z, which is Fraenkel von Neumann's modification of Zermelo's axiom system is described here. The chief motivation in formulating the axioms of system Z is the elimination of set theoretical antinomies. Although Z is sufficient for developing classical set theory, one weakness is that its axioms of subsets and substitution are not proper axioms, but axiom schemas. Also Z imposes some undesirable restrictions on the set concept. The more recent formulations by von Neumann, Bernays and Gödel, of the axiomatization of set theory have been attempts to overcome these shortcomings.

2. *Matrix theorems with application to the rate behavior of metabolic systems*, by Dr. J. Z. Hearon, Office of Mathematical Research, National Institute of Arthritis and Metabolic Diseases, introduced by the Secretary.

For a linear physical system with rate matrix K, the principle of detailed balance insures the existence of a positive definite diagonal matrix A such that KA is symmetric. This implies that K is similar to a symmetric matrix. The consequence of these properties are discussed in terms of symmetry properties of the matrix of principle solutions and certain boundary value problems. For a nonlinear system the corresponding properties of the Jacobian matrix are established and discussed in terms of "relaxation time analysis" of systems close to equilibrium.

3. Panel Discussion by Mr. E. C. Douglas, Taft School, representing the Commission on Mathematics.

Since the Commission firmly believes that the study of geometry should continue to be the basis of a full year's work in secondary school mathematics, it has formulated a program for the tenth grade which is directed to the study of geometry. The following are objectives: (1) the acquisition of information about geometric figures in two as well as three dimensions; (2) the development of a better understanding of the deductive system as a way of thinking and acquiring further skill in applying this method to problems in mathematics; (3) the exposure to opportunities for creative and original thinking. The Commission firmly believes that these objectives can be

met by incorporating with plane geometry some coordinate geometry, and essentials of solid geometry and space perception.

4. Panel Discussion by Professor R. J. Walker, Cornell University, representing the School Mathematics Study Group.

Training in clear thinking, precise language, and logical reasoning should appear somewhere in the high school program. Geometry still seems to be the best vehicle to which to attach such training, having more motivation, in its close contact with simple properties of the physical world, than algebra or any non-mathematical subject. Of the several known rigorous developments of Euclidean geometry, the one proposed by G. D. Birkhoff has the advantages of leading rapidly to the more interesting theorems and of making use of the students' knowledge of algebra. Using a modification of the approach, coordinate geometry can be introduced quickly; but the analytic techniques, being largely mechanical in their application are not over-stressed. The two- and three-dimensional aspects of geometry are closely interwoven.

5. Panel Discussion by Professor Walter Prenowitz, Brooklyn College, speaking on *Geometry in the college curriculum.*

JUNE R. JENSEN, *Secretary*

THE APRIL MEETING OF THE MINNESOTA SECTION

The annual spring meeting of the Minnesota Section of the Mathematical Association of America was held on April 25, 1959, at the University of Minnesota in Minneapolis. The meeting was a joint one with the Minnesota Council of Teachers of Mathematics. Professor J. M. H. Olmsted of the University of Minnesota presided at the morning session. The Rev. W. C. Kalinowski, O.S.B., of St. John's University presided at the afternoon session. There were 114 persons in attendance, including 74 members of the Association.

At the business meeting Professor Leon Green of the University of Minnesota reported for the High School Contest Committee on another successful year in giving the Association's high school mathematics contest.

Professor Warren Loud, chairman of the Section's Nominating Committee, nominated the following slate of section officers who were subsequently elected: Chairman, Professor Gerald Heuer of Concordia College; Secretary, Professor F. L. Wolf of Carleton College; Members of the Executive Committee, Professor John Hafstrom of the University of Minnesota, Duluth Branch, Rev. W. C. Kalinowski and Professor James Serrin of the University of Minnesota.

The following papers were presented:

1. *Evaluation of achievement in mathematics,* by Professor P. C. Rosenbloom, University of Minnesota and the Minnesota State Department of Education.

The evaluation of new curricular material is a vast research problem, requiring the cooperation of mathematicians, psychologists, and educators. Existing achievement tests measure only rote mastery of skills. New tests will have to be devised to measure understanding of concepts, powers of generalization and abstraction, interest, attitudes, and appreciations. The organization of the Minnesota National Laboratory for the Improvement of Secondary School Mathematics as an agency of the Minnesota State Department of Education was described.

2. *Panel discussion on introducing "modern" mathematics into the curriculum,* by Professors B. R. Gelbaum, University of Minnesota, Seymour Schuster and R. W. Sloan, Carleton College.

3. *Motivation of secondary school mathematics courses through the modern electronic computer,* by Dr. R. E. Smith, Control Data Corporation.

The essentials of computer programming can be taught to secondary school students. Used

as a special topic it can do the following: 1) open to students the door to the world of practical computations based upon that which the mathematics teachers are now teaching; 2) demonstrate to the student that the "long" and "tiresome" hand method of solving a problem is often the shortest and "nicest" approach for the computer; 3) introduce to the student the need of organization, planning, and logical thinking required to analyze each problem; 4) bring to the student an awareness of the impossible both in capacity and timing; 5) make the student acquainted with the fundamentals of a science which is growing, important in his own state, and may some day claim him as an employee.

4. *Round-off in square root algorithms*, by Dr. D. M. Brown, Remington Rand Division, Sperry Rand Corporation, St. Paul, Minnesota.

Newton's method for solving for the zeros of $y=x^2-B$ provides the algorithm $x_{i+1}=(B/x_i+x_i)/2$ which, for positive x_0, converges to $\sqrt{B}$. Since finding the square root of an integer M, correct to d decimal places can be reduced to finding the root of the integer $(10)^{2d}M$ to the nearest integer; we restrict B to integral values. If we use $t_i=[x_i]$, we find that t_i converges to $\sqrt{B}$, unless $B=C^2-1$, in which case t_i oscillates between C and $C-1$. If we use $x_{i+1}=[(B-1)/x_i+x_i+1]/2$, it can be shown that $[x_i]$ always converges to the nearest integer to B. Hence this last formula provides as good an approximation to $\sqrt{B}$ as one can find.

5. *The Banach-Tarski paradox*, by Professor L. H. Loomis, Harvard University.

A derivation of the Banach-Tarski theorem was given.

6. *A functional inequality* by Dr. George Brauer, University of Minnesota, and Mr. Donald Kurth, Minneapolis-Honeywell Regulator Company, Hopkins Minnesota.

Let $f(x)$ denote a real-valued function and let f^n denote the nth iterate of f. The continuous functions $f(x)$ such that (1) $f^n(x)\geq x$, $-\infty<x<\infty$, for some positive integer n, are characterized here. If n is odd and $f(x)$ is a solution of (1) then $f(x)\geq x$ for all x. If n is even and $f(x)$ is a solution of (1), then either $f(x)\geq x$ for all x or $f^2(x)$ is identically equal to x.

7. *The numerical range of a matrix operator*, by Professor W. S. Loud, University of Minnesota.

The numerical range of a linear operator in a two-dimensional complex vector space is shown to be an ellipse in the convex plane with the two eigenvalues of the operator as foci. If the operator has the matrix $\{a_{ij}\}$, the numerical range is the set of values assumed by $\sum a_{ij}\bar{x}_i x_j$ for all $\{x_1, x_2\}$ with $|x_1|^2+|x_2|^2=1$. When the matrix is reduced to subdiagonal form by a unitary transformation, the quadratic form becomes $\lambda_1|x_1|^2+\lambda_2|x_2|^2+\epsilon x_1\bar{x}_2$, where $\epsilon=0$ if and only if the operator is normal, and $|\epsilon|$ is independent of the diagonalizing unitary transformation. The minor axis of the elipse is $|\epsilon|$.

F. L. WOLF, *Secretary*

THE APRIL MEETING OF THE MISSOURI SECTION

The Missouri Section of the Mathematical Association of America met at Lindenwood College, St. Charles, Missouri, on Saturday, April 25, 1959. Professor H. M. MacNeille, Vice-Chairman of the Section, presided at the morning program session. Professor Francis Regan, Chairman of the Section, presided at the business meeting and the afternoon program session. The total attendance was 56, including 43 members of the Association.

The following officers were elected: Chairman, Professor C. E. Kelly, Central Missouri State College; Vice-Chairman, Professor L. O. Jones, William Jewell College; Secretary-Treasurer, Miss Marian Lesher, Central Missouri State College.

Professor Regan presented the proposed by-laws for the Missouri Section, which were adopted. Mr. Richard Spreckelmeyer, Chairman of the Contest Committee, reported for

that committee. On motion, it was recommended to the new Chairman that Mr. Spreckelmeyer be reappointed to a two year term.

The morning program consisted of a talk by Professor E. E. Moise, University of Michigan, on phases of the work of the School Mathematics Study Group. This was followed by a general discussion.

The afternoon program was held in conjunction with the Missouri Council of Teachers of Mathematics. Professor R. V. Andree, University of Oklahoma, spoke on the subject, *What is this modern mathematics anyway?*

S. LOUISE BEASLEY, *Secretary*

THE APRIL MEETING OF THE NEBRASKA SECTION

The thirty-fifth annual meeting of the Nebraska Section of the Mathematical Association of America was held on April 18, 1959 at the University of Nebraska, Lincoln, Nebraska, in conjunction with the sixty-ninth annual meeting of the Nebraska Academy of Sciences. Professor J. F. Wampler, Chairman of the Section, presided. There were 75 persons present, including 25 members of the Association. The first session, at which two papers were read, was held jointly with the Nebraska Section of the National Council of Teachers of Mathematics.

The following officers were elected for 1959–1960: Chairman, Professor D. W. Miller, University of Nebraska; Vice-Chairman, Professor J. F. Wampler, Nebraska Wesleyan University; Secretary-Treasurer, Professor H. M. Cox, University of Nebraska.

Professor J. M. Earl was continued as Chairman of the Committee on Mathematics Contests. The committee consists of representatives of the Nebraska Section of the Mathematical Association of America, the Nebraska Section of the National Council of Teachers of Mathematics, and of the Nebraska Actuaries Club.

The following papers were presented:

1. *Mathematics curriculum experiments*, by Professor H. L. Prouse, Mankato State College, Mankato, Minnesota.

This year the Mankato State College Campus School has been participating in mathematics curriculum experiments which include: (1) The seventh grade using the School Mathematics Study Group materials; (2) One ninth-grade section using the Illinois First Course materials. Another innovation being tried involves the eighth grade in which a traditional text is used along with supplementary enrichment topics.

Descriptions of the content of the School Mathematics Study Group seventh and eighth grade materials and the Minnesota National Laboratory were given.

2. *The Second Nebraska (Tenth National) Mathematics Contest*, by Professor J. M. Earl, Municipal University of Omaha.

There were 2,443 students from 134 high schools registered for the Contest and 1,862 score sheets were returned for grading. The scores of all of their own contestants were mailed to all competing schools on March 27 together with other materials which included a tally of all team scores and lists of the 20 high team scores and 20 high contestants in both 1958 and 1959. The Contest's new state sponsor, The Nebraska Actuaries Club, has helped greatly with the work and the expense of the Contest, but, again, the main burden has been shouldered by the Bureau of Instructional Research of the University of Nebraska.

3. (By title) *Algebraic factoring: a process of partial addition*, by Mr. G. B. Paulien, Extension Division of the University of Nebraska.

4. *Honors Mathematics Program at the University of Nebraska*, by Professor D. L. Guy, University of Nebraska (read by Professor D. W. Miller).

In the fall of 1958 the University began giving a sequence of three courses for freshman stu-

dents of high calibre. An outline of the selection procedure, content of the course, and a report of results so far is given. In addition, two seminar programs, one for upper classmen and one for beginning graduate students is discussed.

5. *The Riemann zeta-function*, by Mrs. Mildred Gross, University of Nebraska.

The relationship of the Riemann zeta-function to several other arithmetical functions was discussed on an expository basis. The problem of evaluating the function for various values of s was presented, together with a derivation of the value of zeta of 4 using a Fourier series. Also discussed was the analytic continuation of the zeta-function to the entire complex plane and the resulting functional equation which can be used to locate the zeros of the zeta-function. Also, reference was made to Riemann hypothesis.

6. *The concept of function*, by Professor J. F. Wampler, Nebraska Wesleyan University.

This paper consisted of a brief discussion of the historical development of function, remarks concerning functional notation and a comparison of the three main definitions of function. Emphasis was placed upon the role of set theory in the present-day work with functions.

H. M. Cox, *Secretary*

THE APRIL MEETING OF THE OKLAHOMA SECTION

The annual spring meeting of the Oklahoma section of the Mathematical Association of America was held at The University of Tulsa, Tulsa, Oklahoma, on April 10, 1959. Professor D. P. Richardson, Chairman of the Section, presided during the meeting, which was devoted to the reading of mathematical papers of a research or expository nature. There were 70 persons present, including 41 members of the Association.

The following invited addresses of roughly twenty minutes each were presented:

1. *A question on groups and the axioms for the integers*, by Professor J. E. Hoffman, Oklahoma State University.

2. *Runcible groups*, by Mr. A. S. Davis, University of Oklahoma.

A runcible group $G(\cdot)$ consists of a set G and a binary operation $(\cdot)$ satisfying: (1) if $x \cdot y$ is defined and $y \cdot z$ is defined, then either $x \cdot (y \cdot z)$ and $(x \cdot y) \cdot z$ are both undefined, or else both are defined and are equal to the same element in G; (2) G contains an identity element; (3) each element has an inverse in G. Much of group theory, including the Jordan-Hölder theorem, carries over to these "groups without enclosure." If $H(\cdot)$ may be obtained from $G(\cdot)$ by defining products in $G(\cdot)$, then $G(\cdot)$ is said to be groupable into $H(\cdot)$. The class of all runcible groups groupable into a given group forms a complete lattice.

3. *A technique for approaching fixed point theorems*, by Mr. Tetsundo Sekiguchi, Oklahoma State University, introduced by the Secretary.

4. *On computing the fertility of sets of premises in the propositional calculus*, by Professor W. E. Stuermann, University of Tulsa.

This paper examines the condition imposed by a premise conjunction on any conclusion implied by it. The paper then discloses the pattern exhibited by the frequency distribution of all possible truth functions on the basis of the number of clauses in the developed disjunctive normal form—it is Pascal's triangle. This makes it possible to compute the number and frequency distribution of, and to construct, all valid core *conclusions* for a given set of premises. We have, then, a technique for computing the fertility of any given set of premises or the relative fertility of different sets of premises. Examples are given.

5. *The skew cubic*, by Professor N. A. Court, University of Oklahoma.

The skew cubic C_3 may be considered as the locus of the harmonic pole, for a tetrahedron (T) inscribed in C_3 of a variable plane passing through a definite line s. The lines s', s'', s''' which corre-

spond to s in the three skew harmonic homologies having for axes the three pairs of opposite edges of (T) are secants of C, and the four lines s, s', s'', s''' form a hyperbolic group.

The tangents to C_3 at the vertices of (T) meet the respectively opposite faces of (T) in the four vertices of a tetrahedron (U) which forms with (T) a Moebius pair. Given a pair of Moebius tetrahedrons, a skew cubic, and only one, may be circumscribed about each of them so that the lines joining corresponding vertices of the two tetrahedrons shall touch the two cubics at the respective vertices.

6. *Some remarks on indecomposable continua*, by Mr. P. E. Long, Oklahoma State University.

Mr. Long quoted and discussed the definition of an indecomposable continuum as a continuum which is not the union of two of its proper subcontinua, giving several examples and stating two necessary and sufficient conditions that a continuum be indecomposable. He also discussed some recent results in this field.

7. *A characterization of property* (P), by Dr. J. E. Scroggs, University of Arkansas.

A bounded normal operator A on the Hilbert space H is said to have *property* (P) if every invariant subspace of H under A is also reducing. The sequence of operators A_n is said to converge to the operator A in the weak topology for the space of operators on H if $\lim_{n\to\infty} |(A_n x, y)-(Ax, y)| = 0$, for every $x, y \in H$. Wermer has shown that a sufficient (but not necessary) condition that A have property (P) is the weak convergence of a sequence of polynomials in A to A^*. A necessary and sufficient condition that A have property (P) is given.

8. *A Toeplitz endomorphism theorem*, by Mr. E. P. Kelley, Jr., Oklahoma State University.

A proof of the following theorem was given. *If G is a group, H a normal subgroup of G, α an endomorphism on G such that $h\alpha = h$ for $h \in H$, then $H\alpha^{-1} = H \times K$.* An endomorphism having this property is called a Toeplitz endomorphism. If G is the additive group of the residue class ring of bounded sequences of real numbers modulo the ideal of null sequences then the regular or Toeplitz matrices induce an endomorphism on G which is an automorphism on the subgroup of convergent sequences modulo the null sequences. The null sequence ideal is maximal in the subring of convergences and the residue class ring is the real field.

9. *Lipschitzian parameterizations and existence of minima in the calculus of variations*, by Dr. G. M. Ewing, The Army Artillery and Missile School, Fort Sill, Oklahoma.

10. *Abstract summability methods*, by Professor R. B. Deal, Oklahoma State University.

11. *A generalized derivative*, by Mr. T. W. Cairns, Oklahoma State University.

12. *Derivation of equations suitable for a statistical study of the amount of excess propellant in a liquid-bipropellant rocket*, by Mr. R. M. McDonald, University of Tulsa and Douglas Aircraft Company, Tulsa, Oklahoma.

The amount of unused propellant remaining aboard a liquid bipropellant rocket at burnout must be known to load the rocket efficiently. Numerous random influences, however, make it impossible to predict this amount analytically. A statistical evaluation of the rocket's propellant utilization parameters is too difficult to obtain. Therefore, we study artificial stochastic models of the necessary parameters. A mathematical model of the average in-flight propellant mixture ratio is developed. Mention is made of the other parameters which determine propellant utilization.

R. V. ANDREE, *Secretary*

THE APRIL MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the Mathematical Association of

America was held at Arizona State University, Tempe, Arizona, on April 10–11, 1959. Professor J. H. Butchart, Chairman of the Section, presided at the afternoon session on April 10, and also at the morning session on April 11. There were 53 persons in attendance, including 43 members of the Association.

The following officers were elected: Chairman, Professor R. B. Crouch, New Mexico State University; Vice-Chairman, Professor Harvey Cohn, University of Arizona; Secretary-Treasurer, Professor Deonisie Trifan, University of Arizona.

The following papers were presented:

1. *Some conjectures concerning the decomposition of an algebraic integer into squares*, by Professor Harvey Cohn, University of Arizona.

The author made large scale tests to explore the representability of a totally positive integer, as the sum of a given number of quadratic squares; enabling the author to make very many conjectures including a provable conjecture that the integer $a+2b\sqrt{2}$, $a>2b\sqrt{2}\geqq 0$ can always be represented as the sum of four squares in a manner similar to the known case of $\sqrt{5}$ (Götzky, Ann. of Math. vol. 100). In addition, in no case tested from $\sqrt{2}$ to $\sqrt{41}$ were six squares ever needed. Results will appear in Numerical Mathematics, vol. 1. The GEORGE computer at Argonne Laboratories of the AEC was used. Supported by NSF grant No. G-4222.

2. *Orthoptic of the cardioid*, by Professor J. H. Butchart, Arizona State College, Flagstaff.

It is proved synthetically that the orthoptic of a cardioid consists of two branches, one a circle and the other a nodal limaçon.

3. *A theoretical analysis of errors in hyperbolic positioning systems*, by Mr. G. J. Simmons, Sandia Corporation.

The error involved in the use of asymptotic formulas with hyperbolic positioning systems, valid for very distant sources, was investigated for near sources. This error function was expressed in terms of the differential operator $d(x^{-1})/dx$ which generates a set of polynomial coefficients in the series expansion of the error. The computed error surface $\epsilon(\beta, Z)$ as a function of direction and distance was exhibited. Finally this error was shown to be geometrically the angular difference between the radius vector to a point and the asymptote to the hyperbola through the point.

4. *Modernizing the mathematics curriculum*, by Professor Charles Wexler, Arizona State University.

Desirable improvements in the mathematics curriculum in the grade school and high school are weighed against the capabilities of the teachers. (1) Experiments by Beberman, by Rosenbloom, and by the author indicate that the better 30% or 40% of the grade school students can learn the equivalent of one year of algebra and perhaps one year of geometry under the right kind of teaching. NSF Institutes are greatly needed in this area. (2) In the high schools, teachers are more or less ignoring the prodding of the Commission on Mathematics to include set theory, modern algebra, and statistical inference, but instead, and against the words of caution of the Commission and C.U.P., they are enthusiastically including analytic geometry and calculus in accelerated programs. It is suggested that NSF Institutes fall in with this irresistible trend and include refresher work in calculus plus function theory background. (3) In the colleges, the completion of a strong calculus course should take precedence over a course in set theory and modern algebra, although mathematics majors might take such a course concurrently.

5. *Extensions of homomorphisms*, by Professor E. A. Walker, New Mexico State University.

Let G be a reduced Abelian p-group and B a basic subgroup of G. Let α be a homomorphism of B into G. It is shown that α can be extended in at most one way to an endomorphism of G. In particular, an endomorphism of G that leaves a basic subgroup elementwise fixed is the identity

automorphism of G. These results follow from a more general theorem for arbitrary Abelian groups.

6. *Uniform distribution and almost periodic functions*, by Professor G. M. Petersen, University of New Mexico.

A well-known theorem of Weyl states that a sequence is uniformly distributed if and only if for every $h=1, 2, \cdots$, $\lim_{n\to\infty} 1/(n+1)\sum_{k=0}^{n} e(2\pi h s_k)=0$ where we write $e(x)$ for e^{ix}. If $\{s_k\}$ is a sequence, bounded or unbounded, we shall call $\{s_k\}$ uniformly distributed with respect to the sequence $\{\lambda_p\}$ if $\lim_{n\to\infty} 1/(n+1)\sum_{k=0}^{n} e(\lambda_p s_k)=0$ for each p. We then prove the

THEOREM. *Let $f(x)$ be a u.a.p. function with Fourier exponents $\{\lambda_p\}$, then* $\lim_{n\to\infty} 1/(n+1)\sum_{k=0}^{n} f(s_k)=\lim_{T\to\infty}(1/T)\int_0^T f(x)dx$ *if $\{s_k\}$ is uniformly distributed with respect to $\{\lambda_p\}$.*

7. *On ideals in partially ordered groups*, by Professor J. W. P. Mayer-Kalkschmidt, University of New Mexico, introduced by the Secretary.

Let P be a group in which a partial ordering is defined. The author considers the properties (For the notation compare O. Frink, *Ideals in partially ordered sets*, this MONTHLY, vol. 61, 1954, 223–234): P_1: If $x\leq y$, then $axb\leq ayb$; P_2: $a(A^{*+})b=(aAb)^{*+}$; P_3: If J is an ideal then aJb is an ideal; P_4: If J is a completely irreducible ideal, then aJb is a completely irreducible ideal. He shows that the properties P_1, P_2, P_3, P_4 are equivalent, and discusses some implications of this fact. (This research was sponsored by a National Science Foundation Grant.)

8. *Loci associated with families of osculants*, by Mr. Louis Child, New Mexico State University.

At P_0 on a plane curve Γ, a member g_8 of the 6-parameter family G_8 of 8-pointic quartics, meets each member f_8 of the 1-parameter family F_8 of 8-pointic cubics in 8 points at P_0 and ordinarily at 4 additional points, through which passes a unique 4-pointic cubic δ_4 having a node at P_0. The center of curvature Q of the non-tangent branch of δ_4 at P_0 is thus unique for f_8 relative to g_8 so that F_8 determines a curve ϕ of centers for g_8. Each 1-parameter subfamily G of G_8 then defines a 1-parameter set Φ of curves ϕ whose envelope is unique for P_0.

9. *Use of orthogonal functions for least squares approximation with missing points*, by Professor E. L. Walter, New Mexico State University.

The author approximates a function with domain D by a set of functions orthogonal on O, where O has p points and D consists of $p-m$ of these and n additional points. Approximate coefficients are obtained using the $p-m+n$ values over D, then corrected with the solutions of $m+n$ equations. If $m+n<N$, this method is generally easier than the usual one.

10. *On generalized damped oscillations*, by Professor Oswald Wyler, University of New Mexico.

An autonomous system $\dot{x}=f(x, y)$, $\dot{y}=g(x, y)$, with a finite number of critical points, and with a first integral $E(x, y)=$const., represents a system of *generalized undamped oscillations*. A second system $\dot{x}=f_1(x, y)$, $\dot{y}=g_1(x, y)$, with the same critical points, and the property that $E(x, y)$ is decreasing along any orbit of this system, and never constant on a non-critical orbit, represents a system of *generalized damped oscillations*. The present paper discusses the geometric theory of generalized damped oscillations. It is shown, among other things, that every critical point at which $E(x, y)$ does not have a maximum (minimum) is the limit point of at least one noncritical orbit as $t\to+\infty$ $(t\to-\infty)$.

11. *Why and how we should correct the mistakes of Euclid*, by Professor P. H. Daus, University of California, Los Angeles.

12. *The absolute value criterion for curve fitting*, by Mr. R. M. Parker, Air Force Missile Development Center, Holloman Air Force Base, New Mexico.

The problem as to whether there might be advantages in making the sum of the absolute

values of the residuals a minimum in fitting a curve to observed data is investigated. The method is shown to be unsatisfactory because of the difficulty of the procedures involved and because we usually obtain, not a unique solution, but a family of solutions.

13. *Calculation of flux using spherical mirrors*, by Professor J. R. Foote, University of New Mexico.

The basic problem is the optimum manner of replacing paraboloidal surface elements in a large solar furnace by spherical mirror approximations. Since at each point of a paraboloid there are two principal radii of curvature, two spherical approximations are considered. Formulas for both cases are derived by which the flux concentration, resulting from collections of such approximating mirrors, at the paraboloid focus can be calculated. Numerical results for one array indicate that over ninety-seven percent of the performance of a single-surface paraboloid can be obtained, other conditions being equal.

14. *Basic earth satellite orbit*, by Mr. H. W. Burnette, Air Force Missile Development Center, Holloman Air Force Base, New Mexico.

A basic earth satellite orbit which is derived from Kepler's laws is described. Some of the factors involved in the generalization of this orbit are included.

15. *Some new techniques in hydrodynamics*, by Professor M. R. Bottaccini, University of Arizona.

The old problem of the resistance to the motion of a body within an ideal fluid has been solved recently. This solution permits evaluation of force and moment on any Rankine body moving in an arbitrary potential flow. The only difficulty in the application is the need to express the potential in terms of singularities. If, however, the potential can be expanded into a series of Legendre functions, it is possible to transform to a series of singularities by an integral transformation which is a simple extension of Maxwell's theory of poles.

16. *Lagrangian interpolation in information theory*, by Mr. Ali Kyrala, Goodyear Aircraft Corporation, Litchfield Park, Arizona.

The sampling theorems of information theory are shown to be a particular type of Lagrangian interpolation. These theorems are often applied in industry to determine the approximate behavior of functions or their ensemble averages from a *denumerable* set of (sample) values. The method of derivation presented is that of *interpolation by entire functions* which demonstrates clearly the *non-uniqueness* of this type of sampling theorem and indicates what kind of assumptions may be made about the asymptotic behavior of the sampled function in order to secure uniqueness. The connection with spectral behavior is briefly discussed.

17. *Some remarks on irreducible bases for infinite symmetric groups*, by Professor R. B. Crouch and Mr. A. Gray, New Mexico State University.

Let S be a countable set. Let $S(d, d^+)$ be the infinite symmetric group on S. Some normal sets of elements of $S(d, d^+)$ are given. These form bases for $S(d, d^+)$. The problem of reducing to irreducible sets is discussed.

18. *A formularization of some experimental data*, by Dr. C. R. Cassity, New Mexico Institute of Mining and Technology.

From approximately 1000 measurements of an independent variable y as a function of four dependent variables, a formula is derived which satisfactorily represents the data. Standard deviations in the coefficients of the formula and the standard error in the measurements are included.

DEONISIE TRIFAN, *Secretary*

THE MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The annual spring meeting of the Allegheny Mountain Section of the Mathematical Association of America was held on May 2, 1959 at the University of Pittsburgh, Pittsburgh, Pennsylvania. The Section Chairman, Professor I. Dee Peters of West Virginia University, presided at the morning session. Professor J. S. Taylor of the University of Pittsburgh presided at the afternoon session. There were 95 persons registered, including 50 members of the Association.

At the business meeting, the following officers were elected: Chairman, Dr. B. H. Mount, Westinghouse Electric Corporation; Secretary-Treasurer, Professor Evan Johnson, Jr., Pennsylvania State University; Executive Committee, Professor E. F. Myers, University of Pittsburgh and Professor A. B. Cunningham, West Virginia University. Professors E. E. Posey and J. H. Neelley, co-chairmen of the Section Committee on High School Contests, reported that 3811 students, representing 192 high schools, participated in the 1959 contest.

The following short papers were presented:

1. *A new method for the approximate solution of differential equations*, by Dr. Philip Cooperman, University of Pittsburgh, introduced by the Secretary.

The method used here is to bound the solutions of a given differential equation by the solutions of *other* differential equations, the initial values being the same for all of the solutions. It will be shown that this process can be applied to a wide class of linear and nonlinear equations. The approximate solution is then the average of the upper and lower bounds, and the error is less than one-half the differences between the upper and lower bounds. The advantage over numerical methods is that the approximation is given explicitly in terms of elementary functions.

2. *Singularities of three-dimensional harmonic functions*, by Dr. R. P. Gilbert, University of Pittsburgh, introduced by the Secretary.

Bergman has considered the class of harmonic functions, which may be generated by the Whittaker operator from algebraic functions of two complex variables, and has given a simple procedure for obtaining the singularities. In this paper the method is generalized to apply to arbitrary functions of two complex variables. It is found that the possible singularities of the harmonic function lie on the "envelope" of a family of complex surfaces. To locate the actual singularities use is made of an inverse Whittaker operator to see which of the possible singularities correspond to singularities of the function of two complex variables.

3. *Logic for applying topological methods to electric networks*, by Dr. R. W. Long, Westinghouse Electric Corporation, East Pittsburgh, Pennsylvania.

A logic is presented which enables one to form the mesh equilibrium equations for an electrical network from raw system data. The branches of a network graph are divided into two sets designated as link branches and tree branches. Introduced are the concepts of a tree limb and limb nodes. Associating each row of a matrix with a link branch and each column with a tree branch, each term of a column will consist of 0, 1, or -1 depending on whether the particular link branch is contained in the definition of the tree branch associated with the column. Designating this matrix by T, the branch impedance matrix by Z, the link currents vector by I_L, and the impressed voltage vector *by* E, the equilibrium equations are $TZT'I_L=E$.

4. *An integral theorem for harmonic vectors in three variables*, by Professor Josephine M. Mitchell, Pennsylvania State University.

Let $\overrightarrow{H(X)}=(H_1(X), H_2(X), H_3(X))$ be a harmonic vector defined in Euclidean 3-space: $X=(x, y, z)$. The integral formula $H_1(X)=(1/2\pi i)\int_{\mathfrak{L}} f(u, \zeta, S)\zeta^{-1}d\zeta$ holds, where f is a rational function of u, ζ and S, the arguments being connected by an algebraic equation, $u=x+\frac{1}{2}iy(\zeta+\zeta^{-1})+\frac{1}{2}z(\zeta-\zeta^{-1})$ and $\mathfrak{L}$ is a simple closed curve in the ζ-plane. Similar formulas hold for H_2 and H_3. A

relation is obtained between a linear combination of transcendental functions, $\int_{x_{k-1}}^{x_k} \vec{H} \cdot d\vec{X}$, increased by period functions and integrals of certain algebraic functions of one complex variable also increased by period functions. This generalizes a theorem of S. Bergman for rational $f(u, \zeta)$.

5. *Note on a paper of Klamkin*, by Mr. H. W. Gould, West Virginia University.

M. S. Klamkin (this MONTHLY, vol. 64, p. 91) has given a set of formulas which purport to express a generalized geometric series in terms of Stirling numbers of the first kind. It is shown in the present paper that the formulas in question are not valid. It is shown that the problem is equivalent to finding an explicit summation expression giving the Stirling numbers of the second kind in terms of the corresponding numbers of the first kind. One solution to this is exhibited. The converse problem was first solved by Schlömilch (Crelle's Journal, vol. 44, 1852).

6. *High school calculus*, by Professor J. H. Neelley, Carnegie Institute of Technology.

This paper gives the experiences, at Carnegie Institute of Technology for the past two years, with freshmen who have had some calculus before coming to the university. It extends the material given under the title, *What to do about a new kind of freshman*. That paper was presented to the M.A.A. last May and appears in the *Mathematical Education Notes* of this issue, pp. 584–586. The conclusion is that calculus should not be taught in high schools.

7. *A new look at the problem of mathematics teaching*, by Professor Martin Levine, Pennsylvania State University, McKeesport Center.

This presentation dealt with the inadequacy of present methods of counseling for the selection of potential teachers. The major difficulty in teaching mathematics was attributed to the inadequacy of the secondary school faculty in their attitude toward mathematics. Today's methods of counseling students regarding their vocational interest in mathematics teaching was analyzed. Referencing the Strong *Vocational Interest Blank*, the criteria used to determine the successful teacher were questioned and thought to be inadequate. Evaluating a good mathematics teacher's interests represents the area which may yield maximum improvement in mathematics education.

8. *The new mathematics curriculum at Grove City College*, by Professor H. F. Bechtell, Grove City College.

Grove City College recently implemented a mathematics program which it felt would make the most effective use of a minimum number of instructors and still satisfy the needs of the liberal arts and engineering students. A careful analysis had disclosed that the standard courses through the calculus in the first two years and then diversification was the best program. The report is a summary of this analysis and is given in view of the fact that these results contradict some of the recent trends in programs for prospective teachers and students requiring a terminal course.

B. H. MOUNT, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The thirty-eighth annual meeting of the Illinois Section of the Mathematical Association of America was held at Millikin University, Decatur, Illinois, on May 8 and 9, 1959. Professor Arthur Hallerberg, Chairman of the Section, presided at all sessions. There were 79 persons in attendance, including 66 members of the Association, representing 5 high schools and 21 junior colleges, colleges and universities.

Reports were given by the Committee for the Strengthening of the Teaching of Mathematics and the Committee on Contests and Awards. The Section approved a recommendation of Governor-elect Rothwell Stephens that the Chairman appoint a committee to study the role of the Illinois Section in relation to other educational organizations within the state of Illinois. The following officers were elected to serve for the coming year: Chairman, Professor Donald Myers, Millikin University; Vice-Chairman,

Professor Douglas Daly, Illinois Wesleyan University; Secretary-Treasurer, Professor Wayne McGaughey, Bradley University.

The featured speaker following the banquet was Dean W. L. Everitt of the College of Engineering, University of Illinois. He presented his views on Engineering Education in Russia which he formed following a three weeks study and inspection in the U.S.S.R. during December, 1958.

Following a brief welcome by President Paul McKay of Millikin University, the following program was presented:

1. *Reducing levels of abstraction in mathematics for college freshmen*, by Professor Rose Lariviere, University of Illinois, Navy Pier.

The acute necessity of supplying good background workers for high caliber personnel in mathematics and science makes it imperative that the colleges should not delegate the responsibility of developing the less gifted to others. Since the mathematics courses cannot be impoverished, a concerted effort is required in the areas still open to major improvement—the teaching and the text books. To help effect this pedagogical improvement, teachers in action were urged to make suggestions to authors regarding ways of avoiding abstractions and simplifying techniques. A dozen general methods of coping with this problem were given in outline and illustrated. Further specific instances were described in printed material and distributed to the audience.

2. *The teaching of statistics*, by Professor A. L. O'Toole, Western Illinois University.

There still is little statistics instruction in secondary schools; few teachers feel qualified to teach statistics; teacher preparatory institutions have not prepared teachers of statistics; there is no statistics textbook designed for the majority of secondary-school students. In September, 1959, Western Illinois University will inaugurate a program to prepare teachers of statistics. The new program will provide instruction in (1) suitable subject matter of statistics, (2) the objectives of statistics instruction in secondary-schools, and (3) methods and materials of instruction that seem likely to be effective for teaching statistics in high schools and junior colleges.

3. *Equation of a quadric surface through nine points*, by Professor Ruth Rasmusen, Chicago Teachers College.

The problem of writing the equation of a quadric surface passing through nine given points was solved (a) by applying MURT (matrices under row transformations) technique to the solution of nine equations in nine unknowns, and (b) by a generalization for space of the plane geometry problem of writing the equation of a conic section through five points in a plane.

4. *Use of number theory in the curricula*, by Sister Mary Ferrer, Saint Xavier College.

One of the greatest defects of the traditional teaching of mathematics has been the lack of communication between the three levels of learning. The material at the elementary level should not only be teachable but should provide background for high school mathematics as well as for application to other subject matter. The high school mathematics should be a link between elementary mathematics and collegiate mathematics so that progress is consistent.

Number theory supplies a communication between three levels of learning as well as between traditional mathematics and some new mathematical concepts. The concepts of factor, primes and composite numbers, the division algorithm, Euclid's algorithm and congruence were partially developed to show the communication properties and relations of arithmetic, elementary algebra, set theory, groups, rings and fields.

5. *A geometric representation of the tangent of the sum of two angles*, by Professor Larry Wimp, Southern Illinois University.

A geometric construction was made on the unit circle such that by equating different expressions for the length of a line segment one could immediately obtain the identity for the tangent of the sum of two angles, the sum being less than one right angle.

6. *Boolean matrix algebra and linear graphs*, by Professor Franz Hohn, University of Illinois.

In this paper a type of variable linear graph is defined and a Boolean "connection matrix" is associated with each such graph. The algebra of such matrices is then developed and applied to the study of their associated graphs.

7. *Improved programs of instruction in school mathematics*, by Mr. F. B. Allen, Lyons Township High School and Junior College.

This paper reported on the work of the National Council of Teachers of Mathematics Secondary School Curriculum Committee explaining what has been done and the plans for further work during the summer, 1959.

A. W. McGAUGHEY, *Secretary*

THE MAY MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Goucher College, Towson, Maryland, on May 2, 1959. Professor M. W. Oliphant of Georgetown University presided. Ninety-seven members were in attendance.

The following officers were elected for this fiscal year: Chairman, Professor E. E. Floyd, University of Virginia; Vice-chairmen, Professor R. A. Good, University of Maryland, and Dr. Daniel Shanks, David Taylor Model Basin; Secretary, Professor D. B. Lloyd, District of Columbia Teachers College; and Treasurer, Professor T. W. Moore, U. S. Naval Academy.

The section voted a letter of rebuke to be sent to the *School Science and Mathematics* magazine for carrying false advertising about the classical insoluble problems of Greek geometry.

The program included the following presented papers:

1. *On the distribution of prime numbers in arithmetic progressions*, by Dr. Daniel Shanks, David Taylor Model Basin, Washington, D. C.

The question is investigated, mostly empirically, in which sense it is true, that there are more primes $\equiv -1 \pmod 4$ than $\equiv +1 \pmod 4$. If $\Delta(N)$ is this excess (for primes $\leq N$) then $\Delta(N) > 0$; $=0$; and <0 for 2,995,242; 1352; and 3406 values of $N \leq 3 \cdot 10^8$ respectively.

If $\tau(N) = \Delta(N) \cdot \sqrt{N}/\pi(N)$ then $\tau(1000\,k)$ for $1 \leq k \leq 2000$ has a mean value of 1.06 and a distribution between $\tau = -1/8$ and $\tau = 17/8$ which is roughly symmetric around $+1$. Author conjectures that $\operatorname{Lim}_{\pi \to \infty} (1/N) \sum_2^N \tau(n) = +1$. He then shows that this weakness of $4n+1$ is contained solely in $8n+1$ (or $12n+1$) and not at all in $8n+5$ (or $12n+5$). In contrast $10n \pm 1$ are both equally weak relative to $10n \pm 3$. This difference between mod 8 and 12 on the one hand and mod 10 on the other is explained.

2. *Abstract theory of retrieval coding*, by Dr. C. J. Maloney, Mathematics Division, Biological Warfare Laboratories, Ft. Detrick, Maryland.

The quickening pace and burgeoning volume of research and development, much of it government supported, has led to a crisis in the efforts of librarians and documentalists to "control the record." Attempts to apply newer methods, including electronic computers, to this task have emphasized the need for a general theory of the information retrieval process. It is shown herein that the organization of the record by aspect, introduced during the forties by Batten in England and Cordinnear in France, called here "corbat" indexing, requires the minimum extent of memory, *i.e.*, involves zero memory redundancy.

3. *On the calculation, to a high degree of accuracy, of the modified Bessel functions*, by Dr. F. D. Murnaghan, David Taylor Model Basin, Washington, D. C.

If t is any real number, $[\cosh t+2\cosh(t/2)]/3$ is an upper bound, and $[1+2\cosh(3^{-1/2}t/2)]/3$ is a lower bound, for $I_0(t)$. The mean of these two bounds is again an upper bound, and $[\cosh(2^{-1/2}t)+\cosh(t\cos(\pi/12))+\cosh(t\sin(\pi/12))]/3$ is a lower bound, for $I_0(t)$ and, similarly, the mean of these two latter bounds is an upper bound, and $[\cosh(t\cos(\pi/24))+\cosh(t\sin(\pi/24))+\cosh(t\cos(\pi/8))+\cosh(t\sin(\pi/8))+\cosh(t\cos(5\pi/24))+\cosh(t\sin(5\pi/24))]/6$ is a lower bound for $I_0(t)$. The mean of these two latter bounds is greater than $I_0(t)$ and, if $|t|\leq 10$, this excess is less than 10^{-37}. Similar bounds exist for $I_1(t)$.

4. *The number of vertices of a convex polytope*, by Dr. W. W. Jacobs, American University, and Mr. E. D. Schell, Research Center, IBM Corporation, Yorktown Heights, New York.

In Euclidean space of m dimensions, a convex polytope which has n faces of dimension $m-1$ can have at most

$$\binom{n-r}{n-m}+\binom{n-s}{n-m}$$

vertices, where r and s are the greatest integers in $(m+1)/2$ and $(m+2)/2$ respectively. Also for any n and m, it is possible to construct a polytope with the specified number of vertices.

5. *The mathematics of contract bridge*, by Professor J. A. Tierney, U. S. Naval Academy.

A summary was given of problems essentially mathematical in character which arise in contract bridge. This was followed by a study of techniques for solving card combinations. The *law of balanced distribution*, a new principle of outstanding value to declarer, was presented, followed by a modification involving Bayes' Theorem. Several examples were considered of popular misconceptions involving applications of probability to methods of play.

6. *Number theory and computers*, by Dr. Morris Newman, Applied Mathematics Laboratory, National Bureau of Standards, Washington, D. C.

The high speed computer is an invaluable tool for research in number theory and may be used both to discover theorems and to prove them, when this is possible by numerical computation. In this talk some of the recent progress made in analytic number theory concerned with properties of the coefficients of certain elliptic modular forms and depending on machine computation was described. For example, it has been shown that $p(n)$, the number of unrestricted partitions of the positive integer n, fills all residue classes modulo 13 infinitely often, and satisfies congruences modulo 13 of Ramanujan type.

7. *The position of mathematics in South American higher education*, by Mr. Carlos Fallon, Nems-Clarke Company, Silver Spring, Maryland.

Contrary to the Anglo-Saxon version of the romantic Latin, college mathematics in the larger South American universities is taught with rigor and almost Teutonic thoroughness. Rather than the handmaid of science, mathematics is considered the master-science of all technology. Diverse disciplines are grouped into a single *faculty of mathematical, physical, and natural sciences*. On the negative side, the *intuitively obvious* is sometimes made less obvious by an overly rigorous approach, and classical mathematics is taught to such depth that it excludes from the curriculum certain useful modern techniques.

D. B. LLOYD, *Secretary*

THE MAY MEETING OF THE OHIO SECTION

The forty-third annual meeting of the Ohio Section of the Mathematical Association of America was held at Miami University, Oxford, Ohio, on Saturday, May 9, 1959. Professor L. E. Bush, Chairman of the Section, presided at the morning and afternoon

sessions. There were 68 persons registered in attendance, including 55 members of the Association.

Officers elected for the coming year are: Chairman, Professor W. R. Van Voorhis, Fenn College; Secretary-Treasurer, Professor Foster Brooks, Kent State University; Third member of the Executive Committee, Professor E. T. Stapleford, Kent State University. The Program Committee is: Professor R. W. Shoemaker, University of Toledo, Chairman; Professor C. W. Topp, Fenn College; Professor W. E. Restemeyer, University of Cincinnati.

The following papers were presented:

1. *Some problems in teacher training and retraining*, by Professor L. E. Bush, Kent State University.

The speaker raised the question whether some kind of graduate program satisfying the following requirements should be offered for in-service teachers. (1) The program will take the certified mathematics teacher at the level of mathematical competence at which it finds him. (2) It will lead to a degree which will satisfy the school boards' requirements for a salary differential. (3) It will be possible to satisfy the requirements for the degree in the same amount of time as is required by present degrees in other departments which are offered to mathematics teachers as a means of qualifying for the salary differential. This proposal was followed by a discussion period.

2. *The role of a statistics laboratory on a college campus*, by Professor D. R. Whitney, The Ohio State University. (By invitation).

A statistics laboratory on a university campus should be a consulting agency designed to help researchers from any field in statistics. Three phases of help are indicated: in the design of experiments, in the statistical analysis of data, and in the translation of statistical statements into the language of some particular area of research. The benefits of such a laboratory are that: (1) the campus level of competence in scientific research is raised; (2) a source of research-wise and pedagogically interesting problems in statistics is maintained, and (3) students working in the laboratory acquire some much needed experience.

3. *The impact of the electronic computer on the mathematics curriculum*, by Professor R. J. Nelson, Case Institute of Technology. (By invitation).

The electronic computer has so far had slight effect on the mathematics curriculum with the exception of considerably increased interest in numerical analysis and the introduction of courses in programming (if the teacher of programming happens to be a mathematician). The main impact in the next few years will be increased interest in the theory of computers, computability, algorithms, and in the use of computers for mathematical work: theorem proving, testing cases of conjectures, etc. In the curriculum the effect will probably be increased attention to computational and algorithmic mathematics, particularly the theory thereof.

4. *Solutions of the general Monge equation with some extensions*, by Dr. L. V. Robinson, Wright-Patterson Air Force Base, Ohio.

It is shown that the Monge general partial differential equation can be obtained by combining four identities. From this analysis four total differential equations involving three differentials arise. The usual tests can then be applied as to existence of solutions of these and of the existence of any intermediate integrals. How these methods can be extended is indicated.

5. *Review of basic concepts in approximation theory*, by Dr. Gertrude Blanch, Wright Air Development Center, Wright-Patterson Air Force Base, Ohio. (By invitation).

Numbers entering into calculations are usually only approximate and for that reason continuity in the real sense does not exist in numerical processes. Not every construction that is theoretically possible lends itself to numerical construction. Estimates of the size of the error relating the numerical to the theoretical results are difficult to attain. The effort to do so by means of a com-

puter program, introduced by Ramon E. Moore in "range" arithmetic, is a very hopeful step in the right direction. The need for seeking processes that are numerically stable must be emphasized. Finally, some implications of function theoretical concepts were touched upon.

6. *Minimum variance of estimates under stratified sampling*, by Professor W. R. Van Voorhis, Fenn College.

When a population is stratified for purposes of sampling, the variance of the estimated sample mean, x, depends not only upon the allocation of the sample of size n to the several k strata, but also upon the location of x_i, the points of stratification. Necessary and sufficient conditions to yield minimum variance have been established by Dalenius but these conditions do not yield the explicit values of x_i that must be known before an optimum stratification can be made. It is shown that no general proof of the existence of uniqueness is possible. For the case of "proportional" allocation, it is shown that there exists at least one optimal solution for k strata. A method of successive approximations beginning with a first feasible solution is discussed, and examples are given for several well-known distributions.

7. *Mutation view of conics (shadow transformations and primal states)*, by Dr. Beckham Martin, Owens-Illinois Glass Company, Toledo, Ohio.

In the presentation the following salient remarks were made: (A) There had to be a clean break with conventional geometry, which has reached a state of stagnation, before one could ever hope to reach new pinnacles of achievement. (B) Mutation Geometry is the science of intangible change (shadow-transformations). The discussion began with the general conic equation: (1) $Ax^2+Bxy+Cy^2=Dx+Ey+F$. A primal state number P was calculated: (2) $P=2/(\sqrt{B^2+(A-C)^2}+A+C)$ by which (1) was transformed shadow-wise to its primal state: (3) $ax^2+bxy+cy^2=dx+ey+f$ from which the properties of the representative conic may be read off at sight. Example: The eccentricity is given by (4) $e^2=2-(a+c)$

8. *The 1959 mathematical program*, by Professor R. L. Wilson, Ohio Wesleyan University.

A contrast is made between the type of mathematics currently being used in the physical and nonphysical sciences and the type of mathematics so applied a decade or more ago. Implications are drawn for the mathematical education of students in the various fields of specialization. Alternative suggestions for meeting this situation are made.

9. *Composite pattern of primitive Pythagorean triangles formulated by arithmetical progression*, by Mr. R. J. Irwin, Eddie Painton Associates, Inc., Cleveland, Ohio.

The method presented is believed to be easier and quicker to compile than the methods more commonly used. The non-Pythagorean Triangles are eliminated very readily as they follow a rhythmic appearance in these tables. Periodic checks throughout the tables automatically correct preceding calculations. These tables discovered two (probably typographical) errors in existing published tables. The interesting relation of Pythagorean Triangles to prime numbers is also shown.

FOSTER BROOKS, *Secretary*

THE MAY MEETING OF THE ROCKY MOUNTAIN SECTION

The forty-second annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at Utah State University, Logan, Utah, on Friday afternoon and evening and Saturday forenoon, May 8 and 9, 1959. The meeting was divided into several sessions with Professors N. C. Hunsaker, Joe Elich, J. H. Barrett, and Harvey Fletcher presiding. There were 94 persons registered for the meeting, including 60 members of the Association.

Officers elected at the meeting for 1959–1960 were: Chairman, Colonel J. W. Ault,

United States Air Force Academy; Vice-Chairman, Professor L. W. Rutland, University of Colorado; and Secretary-Treasurer, Professor F. M. Carpenter, Colorado School of Mines.

The following papers were presented:

1. *A method of approximating the roots of an equation by quadratic formulae*, by Professor Stephen Kulik, Utah State University.

Two zeros of an analytic function $f(z)$ are approximated with a prescribed accuracy by one application of a quadratic formula. The coefficients of the quadratic equation depend on $D_n(z)$ which is calculated recursively,

$$D_n = f'D_{n-1} - f''fD_{n-2}/2! + \cdots + f^{(n-1)}(-f)^{n-2}D_1/(n-1)! + f^{(n)}(-f)^{n-1}D_0/(n-1)!,$$
$$D_0 = 1, D_1 = f', D_2 = f' - f''f; \quad \text{where} \quad D_n = D_n(z), f^{(n)} = f^{(n)}(z), \ n = 0, 1, \cdots.$$

The roots of the quadratic equation converge to the two zeros of $f(z)$ which are nearer to the number s than the remaining zeros.

2. *A note on a simple matrix isomorphism*, by Professor D. W. Robinson, Brigham Young University.

Let C be the field of complex numbers. Let $\phi: \alpha\phi = A$ be the well-known ring isomorphism of C onto a real subsystem of 2-by-2 matrices over C. Let $f(x)$ be a polynomial over C. It is shown that $\phi(f(\alpha)) = f(\phi(\alpha))$ if and only if $f(\bar{\alpha}) = \overline{f(\alpha)}$, where $\bar{\alpha}$ is the complex conjugate of α. This result is then generalized by considering (1) a ring isomorphism of the n-by-n matrices over C onto a real subsystem of the $2n$-by-$2n$ matrices over C, and (2) functions of matrices.

3. *Definition of "plus" and "times" for the natural numbers*, by Mrs. Jean J. Pederson, Olympus Senior High School of Salt Lake City and University of Utah.

A function may be defined as a set of ordered pairs such that no two distinct pairs of the set have the same first element. Introducing the natural numbers according to the technique of Peano, explicit definitions, as sets of ordered pairs, may then be exhibited for the addition and multiplication functions as applied to the natural numbers.

4. *An extreme value problem for honor students*, by Captain R. C. Rounding and Captain R. L. Eisenman, United States Air Force Academy.

This paper explores the relationship between a problem in which the volume of a cylindrical solid is given and the relative dimensions are to be found to minimize surface area, and the corresponding problem of a rectangular region wherein the area is constant and the perimeter is to be minimized. It is presented as a problem for Honor Students as an example of a technique in mathematical research.

5. *The multiplicity and positiveness of the characteristic numbers of second order Sturm-Liouville systems involving generalized boundary conditions*, by Professor L. C. Barrett, South Dakota School of Mines and Technology.

This paper is concerned with Sturm-Liouville systems that possess boundary conditions involving left and right hand limits of the dependent variable and its first derivative at one or more interior points of a given fundamental interval. Two theorems are cited which provide introductory information concerning the nature of the characteristic functions of such systems. A third theorem describes the orthogonality of the characteristic functions. It is then shown that the characteristic numbers are simple roots of the characteristic equation. Sufficient conditions that these characteristic values be non-negative are also given.

6. *Distribution of zeros of solutions of complex differential equations*, by Mr. N. H. Mines, University of Utah.

A zero-free region of a nontrivial solution of the complex differential equation $[K(z)W']'$

$+G(z)W=0$ for $K(z)\equiv 1$ was obtained by E. Hille (*Transactions American Mathematical Society*, vol. 23, 1922). The same method can be used to obtain a zero free region of a nontrivial solution of the general equation requiring only analyticity of the coefficients and $K(z)\neq 0$.

7. *A vector solution of simultaneous linear equations*, by Professor C. A. Grimm, South Dakota School of Mines and Technology.

Three points in the plane, $Ax+By+Cz=D$, $D\neq 0$, are sufficient to determine, to a scalar multiple, A, B, C. The vector $[A, B, C]$ is orthogonal to the plane, and is a scalar multiple of the cross product of two vectors in the plane determined by the three points in the plane. By generalizing this idea a method is developed by which any set of n nonhomogeneous linear equations in n variables may be solved by evaluating one n by n determinant.

8. *Homogeneous production function*, by Professor E. A. Davis, University of Utah.

9. *Lattice points*, by Professor T. M. Apostol, California Institute of Technology (Lecture sponsored by Mathematical Association of America).

10. *The Caltech experiment in calculus*, by Professor T. M. Apostol, California Institute of Technology. (Invited Address).

11. *An undergraduate course on the topology of a line*, by Professor C. E. Burgess, University of Utah. (Invited Address).

The author described an introductory undergraduate topology course which is based upon axioms that describe a linearly ordered, separable, connected space with no first point and no last point. Such a course has been offered at the University of Utah each year for the last several years. A similar address was given before the Wisconsin Section at Whitewater, Wisconsin, May 11, 1957 (this MONTHLY, vol. 64, 1957, p. 627).

12. *Particular solutions for nonhomogeneous, linear, ordinary difference equations*, by Professors Forrest Dristy and L. C. Barrett, South Dakota School of Mines and Technology, presented by Professor Dristy.

In this paper an identity is derived which relates adjoint difference expressions in much the same way that Lagrange's identity of differential equation theory relates adjoint differential expressions. It is then shown how this identity may be used to determine a particular solution of a nonhomogeneous linear ordinary difference equation once the complementary function is known. Thus, the method provides an alternative to the familiar method of variation of parameters.

13. *Application of Fourier series to difference equations*, by Lieutenant J. N. Christiansen, United States Air Force Academy.

A method for obtaining general solutions to linear difference—differential equations is presented. The method is applied to a simple example and the solution is plotted for a special case. The method presented is valuable in that it requires no knowledge of mathematics beyond that usually gained from a course in advanced calculus. It is easy to apply and reduce the solution to quadratures in a very few steps. Also, the solution contains the initial conditions explicitly. The method is analogous to the use of the Fourier transform for finding solutions to partial differential equations.

14. *Geometrical motivations for determinant type proofs of mean value theorems*, by Mr. R. A. Jacobson and Professor L. C. Barrett, South Dakota School of Mines and Technology, presented by Mr. Jacobson.

The usual proofs of the mean value theorems involve the process of applying Rolle's Theorem to functions or determinants happily designed to yield the desired conclusions. The determinants thus employed are usually introduced without comment. In this paper it is shown that these determinants may be motivated by an analysis originating in a geometrical setting.

15. *An approach to the foundations of intuitionism*, by Mr. David Drake, University of Colorado.

The concept of a formal proof was adapted to intuitionist philosophy by replacing axioms and primitive rules of inference with other assertability criteria, such as metamathematical observation. The primitive symbols of logic were defined by means of words having reference to finite and perceptually concrete situations. An axiom of formalized intuitionist logic was then derived—for the given interpretation of symbols—by the modified proof method.

16. *The program of advanced placement in mathematics at Colorado State University*, by Professor F. M. Stein, Colorado State University.

Many students enter Colorado State University with more than the minimum background to enter the regular sequence of mathematics courses. This is first an outline of the method used to select those in this group who could start the sequence at an advanced level, and second a report on the success of the program thus far.

17. *A method of solving Diophantine quadratic equations*, by Professor B. W. Jones, University of Colorado.

Here a method, originating in some ideas of Edgar Emerson, is given for finding all the integral solutions of certain equations of the form $ax^2-by^2=c$ where a, b, and c are positive integers, given a finite number of such solutions. Geometrically this is equivalent to use of a zigzag pattern of lines. This applies also to some quadratic indefinite forms with cross products. In some respects this method is an improvement over the traditional use of the Pell equation.

F. M. CARPENTER, *Secretary*

THE MAY MEETING OF THE WISCONSIN SECTION

The twenty-seventh annual meeting of the Wisconsin Section of the Mathematical Association of America was held on May 2, 1959, at Wisconsin State College, Platteville, Wisconsin, Professor J. V. Finch, Chairman, presiding. There were 69 present, including 33 members of the Association and 30 members of the Wisconsin Mathematics Council, 15 of whom are also members of the Association.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor C. B. Hanneken, Marquette University; Vice-Chairman, Professor Henry Van Engen, University of Wisconsin; Secretary-Treasurer, Sister Mary Felice, S.S.N.D., Mount Mary College.

The following report of the Section's fourth annual mathematics contest for high school students was given by the Contest Committee Chairman, Professor Earl Swokowski, Marquette University:

For the second consecutive year a preliminary contest examination consisting of multiple-choice questions was given for the purpose of enabling the teachers to better select the participants in the final contest. This examination was given to 12,600 students in 283 high schools on February 26, 1959. The top contestant in each school was awarded a certificate.

The final contest was held on April 11, in 27 centers distributed throughout the State, with 900 participating from 187 schools. Cash prizes of $50, $25, $10, and $1 were awarded to each student in the top 15 percent of the contestants, divided into four groups respectively. In addition initialed M.A.A. award pins were given to the twenty in the top two groups and a certificate to the rest of these groups.

After some discussion during which various suggestions were offered for subsequent contests, the Section Chairman was instructed to continue to carry on the contest as in the past two years.

After an address of welcome by Professor Bjarne Ullsvik, President of Wisconsin

State College, Platteville, the following program was given:

1. *Introduction of algebra through inequalities*, by Mr. Dawson Trine, Wisconsin State College, Platteville, introduced by the Chairman.

This paper was a report from a member of the first year-long Academic Institute of the National Science Foundation held at the University of Wisconsin. It explained a method used the following year to introduce a first course in algebra. The method was basically that of graphing an equation and its two corresponding inequalities, using a number line for each statement. Example problems were given that could be used to introduce various sections of algebra.

2. *Stokes' theorem*, by Professor L. M. Milne-Thomson, Mathematics Research Center, U. S. Army, Madison, Wisconsin, introduced by the Chairman.

It was shown that any function in the xy-plane can be expressed as a function of the complex variable and its conjugate. This leads to a plane formulation of Stokes' theorem (first published by the speaker in 1938), which has numerous applications including the calculation of area, centroid and moments of inertia. The mapping of the cardioid on the unit circle was found and used to illustrate the applications.

3. *Mathematics of ballistic missile guidance*, by Mr. A. J. Pejsa, A. C. Spark Plug Division, General Motors Corporation, Milwaukee, Wisconsin.

The trajectory of a body in the earth's gravitational field is treated by first deriving equations for the case of a nonrotating spherical earth and then superimposing onto them the effects of earth rotation, oblateness, and other lesser nomalies. Error sensitivities indicating the effects of perturbing the various parameters are considered.

A guidance scheme is suggested which would be accurate enough to guide an IREM with tolerable accuracy, using a comparatively simple airborne guidance computer.

4. *The 37th annual meeting of N.C.T.M.*, by Mr. Vincent Brunner, Nicolet High School, Milwaukee, introduced by the Chairman.

The various sections of this meeting greatly emphasized the changes taking place in high school mathematics. These changes involve new methods, new curricula, and new materials. In order to meet the responsibilities imposed on the teachers by these changes, the teachers were urged to take advantage of National Science Foundation Fellowships, in-service training programs, and programs sponsored by industry in an effort to improve background and increase proficiency. Sources of information which are available can be found in the reports of the National Council of Teachers of Mathematics, the Commission of the College Entrance Examination Board, University of Illinois Mathematics Project, University of Maryland Mathematics Project, and other studies.

5. *Mathematics in Sweden*, by Dr. L. F. Wahlstrom, Wisconsin State College, Eau Claire.

The first-year mathematics student at the university in Sweden may take as much as 15 hours of instruction in mathematics alone, and he may spend as much as 45 hours per week on the study of the subject outside of class. Mathematics may be the only subject he is studying during his first year. Although lecture classes may run as large as 200 students or more, small supervised study groups help the student to master the material. Written examinations last four to five hours and are given periodically. The student takes the examination when he is ready for it. If he passes the written examination, he then takes an oral one.

6. *Large section classes in mathematics*, by the Reverend L. J. Heider, S.J., Marquette University.

SISTER MARY FELICE, *Secretary*

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CALENDAR OF FUTURE MEETINGS

Fortieth Summer Meeting, University of Utah, Salt Lake City, Utah, August 31–September 3, 1959.

Forty-third Annual Meeting, Conrad Hilton Hotel, Chicago, Illinois, January 28–30, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Grove City College, Grove City, Pennsylvania, April 30, 1960.

ILLINOIS, Illinois Wesleyan University, Bloomington, May 13–14, 1960.

INDIANA

IOWA, State University of Iowa, Iowa City, October 16, 1959.

KANSAS, Kansas State College of Pittsburg, April 30, 1960.

KENTUCKY, University of Kentucky, Lexington, April, 1960.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, American University, Washington, D. C., December 5, 1959.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 26, 1960.

MINNESOTA

MISSOURI, Central Missouri State College, Warrensburg, April 23, 1960.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1960.

NEW JERSEY, Princeton University, November 7, 1959.

NORTHEASTERN, Boston College, Chestnut Hill, Massachusetts, November 28, 1959.

NORTHERN CALIFORNIA, University of California, Berkeley, January 16, 1960.

OHIO, Kent State University, May 7, 1960.

OKLAHOMA, Oklahoma City University, October 23, 1959.

PACIFIC NORTHWEST, State University of Montana, Missoula, June 17, 1960.

PHILADELPHIA, University of Delaware, Newark, November 28, 1959.

ROCKY MOUNTAIN, United States Air Force Academy, Colorado Springs, Spring, 1960.

SOUTHEASTERN, University of South Carolina, Columbia, April 1–2, 1960.

SOUTHERN CALIFORNIA, Los Angeles State College, March 12, 1960.

SOUTHWESTERN, Air Force Missile Development Center, Holloman Air Force Base, New Mexico, April, 1960.

TEXAS, San Antonio College, April, 1960.

UPPER NEW YORK STATE, University of Rochester, May 7, 1960.

WISCONSIN, Mount Mary College, Milwaukee, May 7, 1960.

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These symposia were held under the auspices of the American Mathematical Society and other interested organizations. The Society itself published the first two volumes. The McGraw-Hill Book Company, Inc., published and sold Numbers 3 through 8. These six volumes have now been transferred to the American Mathematical Society by special arrangement with the McGraw-Hill Book Company, Inc. Orders should be placed through the Society.

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vi+303 pp. Price: $9.75

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v+104 pp. Price: $5.00

Vol. 8. *Calculus of variations and its applications*, 1958.
v+153 pp. Price: $7.50

Vol. 9. *Orbit theory*. Editor R. E. Langer. In preparation.

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Three Notre Dame authors, **Haaser, LaSalle,** and **Sullivan,** spent nearly ten long years writing a two-volume college text called **A COURSE IN MATHEMATICAL ANALYSIS.** Volume I, **An Introduction to Analysis,** was published last May.

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INTRODUCTION TO MATHEMATICAL STATISTICS

ROBERT V. HOGG, Associate Professor of Mathematics
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